Exercise 6

A Sigma Notation and Summation

1
Solution

(a) $1\times 3+2\times 5+3\times 7+4\times 9+5\times 11$

$$=\sum_{r=1}^{5}(r)(2r+1)$$

(b) $\frac{2}{1.2.3}+\frac{3}{3.4.5}+\frac{4}{5.6.7}+\ldots\ldots(2n\text{ terms})$

$$=\frac{1+1}{(1)\times(1+1)\times(1\times 2)}+\frac{2+1}{(2)\times(2+1)\times(2\times 2)}+\frac{3+1}{(2)\times(2+1)\times(2\times 2)}.....+\frac{2n+1}{(2n)\times(2n+1)\times(2n\times 2)}$$

$$=\sum_{r=1}^{2n}\frac{r+1}{r(r+1)(r+2)}$$

(c) $-\frac{1}{1^2}+\frac{1}{3^2}-\frac{1}{5^2}+\frac{1}{7^2}-\frac{1}{9^2}+\ldots\ldots(n\text{ terms})$

$$=-\frac{1}{1^2}+\frac{1}{3^2}-\frac{1}{5^2}+\frac{1}{7^2}-\frac{1}{9^2}+\ldots\ldots+(-1)^n\frac{1}{n^2}$$

$$=(-1)^1\frac{1}{1^2}+(-1)^2\frac{1}{3^2}+(-1)^3\frac{1}{5^2}+(-1)^4\frac{1}{7^2}+(-1)^5\frac{1}{9^2}+\ldots\ldots+(-1)^n\frac{1}{n^2}$$

$$=\sum_{r=1}^{n}(-1)^r\frac{1}{(2r-1)^2}$$

(d) $1.2!+2.3!+3.4!+\ldots\ldots+(n+1\text{ terms})$

$$=1.2!+2.3!+3.4!+\ldots\ldots+(n+1)[(n+1)+1)]!$$

$$=\sum_{r=1}^{n+1}(r)(r+1)!$$

2
Solution

(a) $\sum_{r=1}^{5} \frac{1}{(r+1)!} = \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!}$

(b) $\sum_{r=0}^{10} 2^r = 1 + 2 + 2^2 + \ldots + 2^{10}$

(c) $$\begin{aligned}\sum_{r=1}^{4} \frac{1}{r(r+2)}(-1)^r &= -\frac{1}{1(3)} + \frac{1}{2(4)} - \frac{1}{3(5)} + \frac{1}{4(6)} \\ &= -\frac{1}{3} + \frac{1}{6} - \frac{1}{15} + \frac{1}{24}\end{aligned}$$

3
Solution

(a) $\sum_{r=1}^{n}(3r^2-3r+2)$

$=3\sum_{r=1}^{n}r^2-3\sum_{r=1}^{n}r+2n$

$=\frac{3n}{6}(n+1)(2n+1)-\frac{3n}{2}(n+1)+2n$

$=\frac{n}{2}\left[2n^2-2+4\right]$

$=n(n^2+1)$

Learning point:
Use the results

$\sum_{r=1}^{n}r^2=\frac{1}{6}n(n+1)(2n+1)$ $\quad\quad$ $\sum_{r=1}^{n}r=\frac{n}{2}(n+1)$

(b) $\sum_{r=1}^{n}\left[r(3r-1)\right]$

$=\sum_{r=1}^{n}3r^2-\sum_{r=1}^{n}r$

$=3\left(\frac{n}{6}\right)(n+1)(2n+1)-\frac{n}{2}(n+1)$

$=\frac{n}{2}(n+1)(2n+1-1)$

$=n^2(n+1)$

(c) $\sum_{r=11}^{50}(r^3-2)$

$=\sum_{r=11}^{50}r^3-\sum_{r=11}^{50}2$

$=\sum_{r=1}^{50}r^3-\sum_{r=1}^{10}r^3-\sum_{r=11}^{50}2$

$=\left[\frac{50(1+50)}{2}\right]^2-\left[\frac{10(1+10)}{2}\right]^2-(50-11+1)\times 2$

$=162250-3025-80$

$=1622520$

Learning point:
Recall results

$\sum_{r=m}^{n}r=\sum_{r=1}^{n}r-\sum_{r=1}^{m-1}r,$ where $m\le n.$ $\quad\quad$ $\sum_{r=n}^{n}a=an,$ where a is constant.

(d) $\sum_{r=15}^{100}(3^r - 3^{r-1}) = \sum_{r=15}^{100} 3^{r-1}(3-1)$

$$= \sum_{r=15}^{100} 3^r 3^{-1}(3-1)$$

$$= \frac{2}{3}\sum_{r=15}^{100} 3^r$$

$$= \frac{2}{3}\left[3^{15} + 3^{16} + 3^{17} + \ldots + 3^{100}\right]$$

$$= \frac{2}{3}\left[\frac{3^{15}(3^{100-15+1}-1)}{3-1}\right]$$

$$= \frac{2}{3}\left[\frac{3^{15}(3^{86}-1)}{2}\right]$$

$$= \frac{1}{3}(3^{101} - 3^{15})$$

$$= 3^{100} - 3^{14}$$

Learning point:

Recall result

$\sum_{r=1}^{n} a^r = \frac{a(r^n - 1)}{r-1}$ ⊲ sum of first *n*th of geometric progression

(e) $\sum_{r=2}^{\infty}\left(\frac{1}{5}\right)^r = \left(\frac{1}{5}\right)^2 + \left(\frac{1}{5}\right)^3 + \left(\frac{1}{5}\right)^4 \ldots$

$$= \frac{\left(\frac{1}{5}\right)^2}{1-\frac{1}{5}}$$

$$= \frac{1}{20}$$

Learning point:

Recall result

$\sum_{r=1}^{\infty} a^r = \frac{a}{r-1}$ ⊲ sum to infinity of geometric progression

(f) $\sum_{r=3}^{n}(3n-r)$

$= \sum_{r=3}^{n} 3n - \sum_{r=3}^{n} r$

$= (n-3+1)(3n) - \frac{(n-3+1)}{2}(n+3)$

$= 3n(n-2) - \frac{(n-2)}{2}(n+3)$ $\triangleleft$ factor out $\frac{(n-2)}{2}$

$= \frac{(n-2)}{2}[6n-(n+3)]$

$= \frac{(n-2)}{2}(6n-n-3)$

$= \frac{(n-2)(5n-3)}{2}$

(g) $\sum_{r=-n}^{n}(r^2+6r+8)$

$= \sum_{r=-n}^{n} r^2 + 6\sum_{r=-n}^{n} r + \sum_{r=-n}^{n} 8$

$= 2\sum_{r=1}^{n} r^2 + 6(0) + 8(2n+1)$

$= \frac{2n}{6}(n+1)(2n+1) + 8(2n+1)$

$= (2n+1)\left[\frac{n}{3}(n+1)+8\right]$

(h) $\sum_{r=1}^{n}(\ln 2^{r+1} - r(r+1) + 2^r)$

$= \sum_{r=1}^{n} \ln 2^{r+1} - \sum_{r=1}^{n} r(r+1) + \sum_{r=1}^{n} 2^r$

$= \sum_{r=1}^{n}(r+1)\ln 2 - \sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} r + \sum_{r=1}^{n} 2^r$

$= \ln 2\sum_{r=1}^{n} r + \sum_{r=1}^{n} \ln 2 - \sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} r + \sum_{r=1}^{n} 2^r$

$= (\ln 2 - 1)\sum_{r=1}^{n} r - \sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} 2^r + \sum_{r=1}^{n} \ln 2$

$= (\ln 2 - 1)\left(\frac{n(n+1)}{2}\right) - \left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{2(2^n-1)}{2-1} + n\ln 2$

$= (\ln 2 - 1)\left(\frac{n(n+1)}{2}\right) - \left(\frac{n(n+1)(2n+1)}{6}\right) + 2(2^n-1) + n\ln 2$

4
Solution

(a) Given $\sum_{r=1}^{n} u_r = n^2 + 2$

As $n \to \infty,\ n^2 + 1 \to \infty.$

$\therefore$ the series diverges.

(b) Given $\sum_{r=1}^{n} u_r = \frac{n+1}{n+2}$

$$\sum_{r=1}^{n} u_r = \frac{\frac{n}{n}+\frac{1}{n}}{\frac{n}{n}+\frac{2}{n}} = \frac{1+\frac{1}{n}}{1+\frac{2}{n}}$$ ◁ divide each term by n

As $n \to \infty,\ \frac{1}{n}, \frac{2}{n} \to 0.$

$$\sum_{r=1}^{\infty} u_r = \frac{1+0}{1+0}$$
$$= 1$$

$\therefore$ the series converges.
The sum to infinity is 1.

Alternative way of presentation

$$\sum_{r=1}^{n} u_r = \frac{n+1}{n+2}$$

$$\lim_{n\to\infty} \sum_{r=1}^{n} u_r = \lim_{n\to\infty}\left(\frac{n+1}{n+2}\right) = \lim_{n\to\infty}\left(1-\frac{1}{n+2}\right) = 1$$

$\therefore$ the series converges.

Sum to infinity $= \sum_{r=1}^{\infty} u_r = 1.$

(c) Given $\sum_{r=1}^{n} u_r = \frac{n^2+1}{n+2}$

$$\sum_{r=1}^{n} u_r = \frac{\frac{n^2}{n}+\frac{1}{n}}{\frac{n}{n}+\frac{2}{n}} = \frac{n+\frac{1}{n}}{1+\frac{2}{n}}$$ ◁ divide each term by n

As $n \to \infty,\ \frac{n+\frac{1}{n}}{1+\frac{2}{n}} \to \infty$

$\therefore$ the series diverges.

(d) Given $\sum_{r=1}^{n} u_r = \frac{(3n+1)(n-2)}{(n+2)(2n-3)}$

$$\sum_{r=1}^{n} u_r = \left(\frac{\left(3+\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\left(1+\frac{2}{n}\right)\left(2-\frac{3}{n}\right)} \right)$$ ◁ divide n to each term

As $n \to \infty$, $\frac{1}{n}, \frac{2}{n}, \frac{3}{n} \to 0$.

$$\sum_{r=1}^{\infty} u_r = \left(\frac{(3+0)(1-0)}{(1+0)(2-0)} \right)$$

$$= \frac{3(1)}{1(2)}$$

$$= \frac{3}{2}$$

$\therefore$ the series converges.

$$\sum_{r=1}^{\infty} u_r = \frac{3}{2}$$

Alternative way of presentation

$$\sum_{r=1}^{n} u_r = \frac{(3n+1)(n-2)}{(n+2)(2n-3)}$$

$$\lim_{n\to\infty} \sum_{r=1}^{n} u_r = \lim_{n\to\infty} \left(\frac{(3n+1)(n-2)}{(n+2)(2n-3)} \right) = \lim_{n\to\infty} \left(\frac{\left(3+\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\left(1+\frac{2}{n}\right)\left(2-\frac{3}{n}\right)} \right) = \frac{3(1)}{1(2)} = \frac{3}{2}$$

$\therefore$ the series converges.

The sum to infinity is $\frac{3}{2}$.

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Solution

(a)(i)

$$\sum_{r=0}^{r=n}\frac{1}{(r+1)(r+2)(r+3)}=\sum_{r-1=0}^{r-1=n}\frac{1}{(r)(r+1)(r+2)} \qquad \triangleleft \text{ replace } r \text{ by } r-1$$

$$=\sum_{r=1}^{n+1}\frac{1}{(r)(r+1)(r+2)}$$

$$=\frac{1}{4}-\frac{1}{2((n+1)+1)((n+1)+2)} \qquad \triangleleft \text{ substituting } n+1 \text{ into } \frac{1}{4}-\frac{1}{2(n+1)(n+2)}$$

$$=\frac{1}{4}-\frac{1}{2(n+2)(n+3)}$$

(a)(ii)

$$\sum_{r=2}^{n}\frac{1}{(r+1)r(r-1)}=\sum_{r+1=2}^{r+1=n}\frac{1}{(r+2)(r+1)(r)} \qquad \triangleleft \text{ replace } r \text{ by } r+1$$

$$=\sum_{r=1}^{n-1}\frac{1}{(r+2)(r+1)(r)}$$

$$=\frac{1}{4}-\frac{1}{2((n-1)+1)((n-1)+1)} \qquad \triangleleft \text{ substituting } n-1 \text{ into } \frac{1}{4}-\frac{1}{2(n+1)(n+2)}$$

$$=\frac{1}{4}-\frac{1}{2n(n+1)}$$

(b)(i)

$$\sum_{r=1}^{r=n}\frac{2r+5}{(r+2)^2(r+3)^2}=\sum_{r-2=1}^{r-2=n}\frac{2(r-2)+5}{(r)^2(r+1)^2} \qquad \triangleleft \text{ replace } r \text{ by } r-2$$

$$=\sum_{r=3}^{n+2}\frac{2r+1}{(r)^2(r+1)^2}$$

$$=\sum_{r=1}^{n+2}\frac{2r+1}{(r)^2(r+1)^2}-\frac{2(1)+1}{(1)^2(1+1)^2}-\frac{2(2)+1}{(2)^2(2+1)^2}$$

$$=1-\frac{1}{(n+3)^2}-\frac{3}{4}-\frac{5}{36}$$

$$=\frac{1}{9}-\frac{1}{(n+3)^2}$$

(b)(ii)

$$\sum_{r=2}^{n}\frac{2r-1}{r^2(r-1)^2}=\sum_{r+1=2}^{r+1=n}\frac{2(r+1)-1}{(r+1)^2(r)^2} \qquad \triangleleft \text{ replace } r \text{ by } r+1$$

$$=\sum_{r=1}^{n-1}\frac{2r+1}{(r)^2(r+1)^2}$$

$$=1-\frac{1}{(n-1+1)^2} \qquad \triangleleft \text{ replace } n \text{ by } n-1 \text{ in } 1-\frac{1}{(n+1)^2}$$

$$=1-\frac{1}{n^2}$$

6
Solution

(a) $\sum_{r=0}^{k}(2r+5-3^{1-r})$

$=\sum_{r=0}^{k}(2r+5)-\sum_{r=0}^{k}3^{1-r}$

$=\frac{k+1}{2}(5+(2k+5))-\left[3^{0}+3^{1}+3^{2}+...+3^{1-k}\right]$

$=\frac{k+1}{2}(k+5)-\frac{3\left(1-\left(\frac{1}{3}\right)^{k+1}\right)}{1-\frac{1}{3}}$

$=(k+1)(k+5)-\frac{9}{2}\left(1-\left(\frac{1}{3}\right)^{k+1}\right)$

(b)

$\sum_{r=0}^{r=k}(2r+9-3^{-r-1})=\sum_{r-2=0}^{r-2=k}(2r+9-3^{-r-1})$ ⊲ replace r by $r-2$

$=\sum_{r=2}^{k+2}(2(r-2)+9-3^{-(r-2)-1})$

$=\sum_{r=2}^{k+2}(2r+5-3^{1-r})$

$=\sum_{r=0}^{k+2}(2r+5-3^{1-r})-(2\times 0+5-3^{1-0})-(2\times 1+5-3^{1-1})$

$=((k+2)+1)((k+2)+5)-\frac{9}{2}\left(1-\left(\frac{1}{3}\right)^{k+2+1}\right)-(5-3^{1})-(7-3^{0})$

$=(k+3)(k+7)-\frac{9}{2}\left(1-\left(\frac{1}{3}\right)^{k+3}\right)-8$

$=(k+3)(k+7)-\frac{9}{2}+\frac{9}{2}\left(\frac{1}{3}\right)^{k+3}-8$

$=(k+3)(k+7)-\frac{25}{2}+\frac{9}{2}\left(\frac{1}{3}\right)^{k+3}$

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7
Solution

$$\sum_{r=n+1}^{2n}(2r+1)^2$$

$$=\sum_{r=n+1}^{2n}(4r^2+4r+1)$$

$$=4\sum_{r=n+1}^{2n}r^2+4\sum_{r=n+1}^{2n}r+\sum_{r=n+1}^{2n}1$$

$$=4\left(\sum_{r=1}^{2n}r^2-\sum_{r=1}^{n}r^2\right)+4\left(\frac{2n-(n+1)+1}{2}(n+1+2n)\right)+(2n-(n+1)+1)$$

$$=4\left(\frac{1}{6}(2n)(2n+1)(4n+1)-\frac{1}{6}n(n+1)(2n+1)\right)+2n(3n+1)+n \quad \triangleleft \text{ factor out } \frac{2}{3}n(2n+1)$$

$$=\frac{2}{3}n(2n+1)\left[2(4n+1)-(n+1)\right]+6n^2+3n$$

$$=\frac{2}{3}n(2n+1)(7n+1)+3n(2n+1) \quad \triangleleft \text{ factor out } \frac{1}{3}n(2n+1)$$

$$=\frac{2}{3}n(2n+1)\left[2(7n+1)+3\times 3\right]$$

$$=\frac{1}{3}n(2n+1)\left[14n+2+9\right]$$

$$=\frac{1}{3}n(2n+1)(14n+11)$$

8
Solution

(a)

$$\sum_{r=1}^{k}\left[\left(-\frac{1}{2}\right)^{r+1}+\ln(r+1)\right]$$

$$=\sum_{r=1}^{k}\left(-\frac{1}{2}\right)^{r+1}+\sum_{r=1}^{k}\ln(r+1)$$

$$=\sum_{r=1}^{k}\left[\left(-\frac{1}{2}\right)^{r+1}\right]+\left[\ln 2+\ln 3+\ln 4+\ldots+\ln(k+1)\right]$$

$$=\sum_{r=1}^{k}\left(-\frac{1}{2}\right)^{r}\left(-\frac{1}{2}\right)+\ln(2\times 3\times 4\times\ldots\times(k+1))$$

$$=\left(-\frac{1}{2}\right)\sum_{r=1}^{k}\left(-\frac{1}{2}\right)^{r}+\ln\left[(k+1)!\right]$$

$$=\left(-\frac{1}{2}\right)\left[\left(-\frac{1}{2}\right)^{1}+\left(-\frac{1}{2}\right)^{2}+\ldots\ldots+\left(-\frac{1}{2}\right)^{k}\right]+\ln\left[(k+1)!\right]$$

$$=\left(-\frac{1}{2}\right)\left[\frac{\left(-\frac{1}{2}\right)^{1}\left(1-\left(-\frac{1}{2}\right)^{k}\right)}{1-\left(-\frac{1}{2}\right)}\right]+\ln\left[(k+1)!\right]$$

$$=\frac{1}{4}\frac{2\left(1-\left(-\frac{1}{2}\right)^{k}\right)}{3}+\ln\left[(k+1)!\right]$$

$$=\frac{1}{6}\left[1-\left(-\frac{1}{2}\right)^{k}\right]+\ln\left[(k+1)!\right]$$

(b) As $k\to\infty$, $\left(-\frac{1}{2}\right)^{k}\to 0$ and $\left[(k+1)!\right]\to\infty$

i.e. $\lim_{k\to\infty}\left\{\frac{1}{6}\left[1-\left(-\frac{1}{2}\right)^{k}\right]\right\}=\frac{1}{6}$ and $\lim_{k\to\infty}\left\{\ln\left[(k+1)!\right]\right\}=\infty$

$\therefore$ the sum to infinity of the series does not exist.

9
Solution

(a) Given $v_n = v_{n-1} + n$

$v_2 = v_1 + 2$

$v_3 = v_2 + 3$
$= v_1 + 2 + 3$

$v_4 = v_3 + 4$
$= v_1 + 2 + 3 + 4$

So, $v_n = v_1 + 2 + 3 + 4 + ... + n$

$$= v_1 + \frac{n-1}{2}(2+n)$$

$$= v_1 + \frac{(n-1)(n+2)}{2}$$

$$= A + \frac{(n-1)(n+2)}{2}$$

$$\therefore \; v_n = A + \frac{(n-1)(n+2)}{2}$$

(b) $$\sum_{r=1}^{n} v_r = \sum_{r=1}^{n}\left(A + \frac{(r-1)(r+2)}{2}\right)$$

$$= \sum_{r=1}^{n}\left(A + \frac{r^2+r-2}{2}\right)$$

$$= \sum_{r=1}^{n} A + \sum_{r=1}^{n}\left(\frac{r^2+r-2}{2}\right)$$

$$= An + \sum_{r=1}^{n}\left(\frac{r^2}{2} + \frac{r}{2} - \frac{2}{2}\right)$$

$$= An + \frac{1}{2}\sum_{r=1}^{n} r^2 + \frac{1}{2}\sum_{r=1}^{n} r - \sum_{r=1}^{n} 1$$

$$= An + \frac{1}{2}\left[\frac{1}{6}n(n+1)(2n+1)\right] + \frac{1}{2}\left[\frac{n}{2}(1+n)\right] - n$$

$$= n(A-1) + \frac{1}{12}n(n+1)(2n+1) + \frac{1}{4}(n)(n+1)$$

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Solution

(a) $a_{n+1} = a_n + ka_{n-1}$ ⊲ substituting $n = 1$

$a_2 = a_1 + ka_0$ ⊲ $a_0 = 2$ $a_1 = 7$ and $a_2 = 11$

$11 = 7 + k(2)$

$k = 2$

(b) Given $a_n = A(2^n) + B(-1)^n + C$ .... (1)

Substitute $n = 1$ into (1)

$2 = A + B + C$ .... (2)

Substitute $n = 2$ into (1)

$7 = 2A - B + C$ .... (3)

Substitute $n = 3$ into (1)

$11 = 4A + B + C$ .... (4)

Using GC to solve (2), (3) and (4)

$A = 3, B = -1, C = 0$

$\therefore$ $A = 3, B = -1$ and $C = 0$

(c) From (1): $a_r = 3(2^r) - (-1)^r$ ⊲ substitute $A = 3, B = -1$ and $C = 0$

$$\sum_{r=1}^{n} a_r$$

$$= \sum_{r=1}^{n}\left[3(2^r) - (-1)^r\right]$$

$$= 3\sum_{r=1}^{n}\left[(2^r)\right] - \sum_{r=1}^{n}(-1)^r$$

$$= 3\left[\frac{2(2^n - 1)}{2 - 1}\right] - \left[(-1)^1 + (-1)^2 + \ldots + (-1)^n\right]$$

$$= 6(2^n - 1) - \frac{(-1)\left[(-1)^n - 1\right]}{-1 - 1}$$

$$= 6(2^n - 1) - \frac{1}{2}\left[(-1)^n - 1\right]$$

$$= 6(2^n) - \frac{1}{2}(-1)^n - \frac{11}{2}$$

Alternative Method

$$\sum_{r=1}^{n} a_r$$

$$= \sum_{r=1}^{n}\left[3(2^r) - (-1)^r\right]$$

When n is odd

$$= 3\left[\frac{2(2^n - 1}{2-1}\right] - (-1)$$

$$= 6(2^n - 1) + 1$$

$$= 6(2^n) - 5$$

$$\sum_{r=1}^{n} a_r$$

$$= \sum_{r=1}^{n}\left[3(2^r) - 1(-1)^r\right]$$

When n is even

$$= 3\left[\frac{2(2^n - 1)}{2-1}\right]$$

$$= 6(2^n - 1)$$

$$\therefore \sum_{r=1}^{n} a_r = 6(2^n) - 5, \text{ where } n = \text{odd}$$

$$\sum_{r=1}^{n} a_r = 6(2^n - 1), \text{ when } n \text{ is even}$$

Exercise 6

B Series

11
Solution

$$\sum_{n=1}^{N}(5^{n-1}+2n)$$

$$=\sum_{n=1}^{N}5^{n-1}+2\sum_{n=1}^{N}n$$

$$=\frac{1(5^{N}-1)}{5-1}+2(\frac{N}{2}(N+1))$$

$$=\frac{(5^{N}-1)}{4}+N(N+1)$$

Learning point:
Expess the general term $5^{n-1}+2n$ into summation.

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Solution

$(n+1)^2+(n+2)^2+(n+3)^2+\ldots+(2n)^2$

$$=\sum_{r=n+1}^{2n} r^2$$

$$=\sum_{r=1}^{2n} r^2-\sum_{r=1}^{n} r^2$$

$$=\frac{2n}{6}(2n+1)(4n+1)-\frac{n}{6}(n+1)(2n+1)$$

$$=\frac{n}{6}(2n+1)\left[2(4n+1)-(n+1)\right]$$

$$=\frac{n}{6}(2n+1)\left[8n+2-n-1\right]$$

$$=\frac{n}{6}(2n+1)(7n+1)$$

Learning point:

Use the result: $\sum_{r=1}^{n} r^2=\frac{n(n+1)(2n+1)}{6}$.

$$=\sum_{r=n+1}^{2n}(2r-1)^2$$

$$=\sum_{r=1}^{2n}(2r-1)^2-\sum_{r=1}^{n}(2r-1)^2$$

$$=\sum_{r=1}^{2n}(4r^2-4r+1)-\sum_{r=1}^{n}(4r^2-4r+1)$$

$$=4\left[\sum_{r=1}^{2n} r^2-\sum_{r=1}^{n} r^2\right]+4\left[\sum_{r=1}^{2n}-r+\sum_{r=1}^{n} r\right]+\sum_{r=1}^{2n}1-\sum_{r=1}^{n}1$$ ◃ use the result from **(a)**: $\sum_{r=1}^{2n} r^2-\sum_{r=1}^{n} r^2=\frac{n}{6}(2n+1)(7n+1)$

$$=4\left[\frac{n}{6}(2n+1)(7n+1)\right]+4\left[-\frac{2n}{2}(1+2n)+\frac{n}{2}(1+n)\right]+2n-n$$

$$=\frac{2n}{3}(2n+1)(7n+1)+\left[-4n(1+2n)+2n(1+n)\right]+2n-n$$

$$=\frac{2n}{3}(2n+1)(7n+1)+(-4n-8n^2+2n+2n^2)+n$$

$$=\frac{2n}{3}(2n+1)(7n+1)+(-2n-6n^2)+n$$

$$=\frac{2n}{3}(2n+1)(7n+1)+6n^2-n$$ ◃ factor out $\frac{n}{3}$

$$=\frac{n}{3}\left[2(2n+1)(7n+1)-18n-3\right]$$

$$=\frac{n}{3}\left[2(14n^2+9n+1)-18n-3\right]$$

$$=\frac{n}{3}(28n^2-1)$$

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Solution

$1^2+5^2+9^2+13^2+\ldots+277^2$ ◁ sum of first 70 terms

$=\sum_{r=0}^{69}(1+4r)^2$

$=1+\sum_{r=1}^{69}\left(1+8r+16r^2\right)$

$=1+69+8\sum_{r=1}^{69}r+16\sum_{r=1}^{69}r^2$

$=70+8\left[\frac{69\left(1+69\right)}{2}\right]+16\left[\frac{(69)(69+1)(2\times 69+1)}{6}\right]$ ◁ use the results: $\sum_{r=1}^{n}r^2=\frac{n(n+1)(2n+1)}{6}$ and $\sum_{r=1}^{n}r=\frac{n(n+1)}{2}$.

$=70+8\left[\frac{69\left(70\right)}{2}\right]+16\left[\frac{(69)(70)(139)}{6}\right]$

$=1809710$

Learning point:

$1^2+5^2+9^2+13^2+..$

From the observation, the base of the terms in the series is arithmetic progression,
i.e. 1 5 9 13
$\therefore$ the general term can be $u_r=1+4r,$ where $r=0,$ 1, 2, ... or $u_r=4r-3,$ where $r=1,$ 2, 3, ...

To obtain the 70th term, substitute $r=69$ into $u_r=1+4r$. $\therefore$ $u_{69}=277$.
Alternatively, substitute $r=70$ into $u_r=4r-3$. $\therefore$ $u_{70}=277$.

Alternative Method

$1^2+5^2+9^2+13^2+\ldots+277^2$

$=\sum_{r=1}^{70}(4r-3)^2$

$=\sum_{r=1}^{70}\left(16r^2-24r+9\right)$

$=16\sum_{r=1}^{70}r^2-24\sum_{r=1}^{70}r+\sum_{r=1}^{70}9$

$=16\left[\frac{1}{6}\left(70\right)\left(70+1\right)\left(2\times 70+1\right)\right]-24\left[\frac{1}{2}\left(1+70\right)\right]+9\times 70$

$=1809710$

14
Solution

(a) $\sum_{r=1}^{n}(3r-1)^2$

$=\sum_{r=1}^{n}(9r^2-6r+1)$

$=9\sum_{r=1}^{n}r^2-6\sum_{r=1}^{n}r+\sum_{r=1}^{n}1$

$=9\left(\dfrac{n(n+1)(2n+1)}{6}\right)-6\left(\dfrac{n}{2}(1+n)\right)+n$

$=\dfrac{3}{2}n(n+1)(2n+1)-3n(1+n)+n$

$=3n(n+1)\left[\dfrac{2n+1}{2}-1\right]+n$

$=3n(n+1)\left[\dfrac{2n+1-2}{2}\right]+n$

$=\dfrac{3}{2}n(n+1)(2n-1)+n$

$\therefore\ p=\dfrac{3}{2},\ q=1$

(b) $2^2+3^3+5^2+6^3+8^2+9^3+\ldots+56^2+57^3+59^2+60^3$

$=(2^2+5^2+8^2+\ldots+56^2+59^2)+(3^3+6^3+9^3+\ldots+57^3+60^3)$

$=\sum_{r=1}^{20}(3r-1)^2+\sum_{r=1}^{20}(3r)^3$ $\quad\triangleleft$ use the result in **(a)**: $\sum_{r=1}^{20}(3r-1)^2=\dfrac{3}{2}n(n+1)(2n-1)+n,$ and replace $n=20$

$=\dfrac{3}{2}(20)(20+1)(2\times 20-1)+3^3\sum_{r=1}^{20}r^3$ $\quad\triangleleft$ use the result: $\sum_{r=1}^{n}r^3=\dfrac{n^2(n+1)^2}{4}$

$=\dfrac{3}{2}(20)(20+1)(2\times 20-1)+20+27\left(\dfrac{(20)^2(20+1)^2}{4}\right)$

$=1215290$

15
Solution

(a) Let rth term $= 2a^r + br^2$ .... (1)

Given the first term is 28, i.e. 1st term = 28

Substitute $r = 1$ into (1).

$\therefore \quad 2a^1 + b(1)^2 = 8$

$2a + b = 8$

Given the second term is 28, i.e. 2nd term = 26

Substitute $r = 2$ into (1).

$\therefore \quad 2a^2 + b(2)^2 = 26$

$2a^2 + 4b = 26$

Using GC to solving (1) and (2) simultaneously.

$a = 3 \quad \text{or} \quad 1 \quad (\text{Rejected, since } \; a > 2)$

Substituting $r = 2$ into $2a + b = 8$.

$\therefore b = 2.$

Hence $a = 3$ and $b = 2$.

(b) Sum of the first $2n$ terms

$$= \sum_{r=1}^{2n} u_r$$

$$= \sum_{r=1}^{2n} \left[2(3)^r + 2r^2 \right]$$

$$= 2\sum_{r=1}^{2n} 3^r + \sum_{r=1}^{2n} 2r^2$$

$$= 2\left[(3)^1 + (3)^2 + (3)^3 + \ldots + (3)^{2n} \right] + 2\sum_{r=1}^{2n} r^2$$

$$= 2\left[\frac{3(3^{2n} - 1)}{3 - 1} \right] + \frac{2(2n)}{6}(2n+1)(2\times 2n + 1)$$

$$= 3(3^{2n} - 1) + \frac{2n}{3}(2n+1)(4n+1)$$

Exercise 6

C Ratio Test

27
Solution

Let $a_n = \dfrac{(-1)^n \pi^{2n}}{(2n)!}$. $\therefore\ a_{n+1} = \dfrac{(-1)^{n+1} \pi^{2(n+1)}}{(2(n+1))!}$

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{(-1)^{n+1}\pi^{2(n+1)}}{(2(n+1))!}}{\dfrac{(-1)^n \pi^{2n}}{(2n)!}}$$

$$= \frac{(-1)^{n+1}\pi^{2n+2}}{(2n+2)!} \times \frac{(2n)!}{(-1)^n \pi^{2n}}$$

$$= \frac{(-1)^n(-1)^1\pi^{2n}\pi^2}{(2n+2)!} \times \frac{(2n)!}{(-1)^n \pi^{2n}}$$

$$= -\frac{(-1)\pi^2(2n)!}{(2n+1)(2n+2)(2n)!}$$

$$= -\frac{\pi^2}{(2n+1)(2n+2)}$$

As $n \to \infty$, $(2n+1)(2n+2) \to \infty$. So, $\dfrac{\pi^2}{(2n+1)(2n+2)} \to 0$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\pi^2}{(2n+1)(2n+2)} = 0 < 1$$

Hence by ratio test, $\displaystyle\sum_{r=0}^{\infty}\frac{(-1)^r\pi^{2r}}{(2r)!}$ converges.

$$\sum_{r=0}^{\infty}\frac{(-1)^r\pi^{2r}}{(2r)!}$$

$$= \frac{(-1)^0\pi^{2(0)}}{(2(0))!} + \frac{(-1)^1\pi^{2(1)}}{(2(1))!} + \frac{(-1)^2\pi^{2(2)}}{(2(2))!} + ...$$

$= 1 - \dfrac{\pi^2}{2!} + \dfrac{\pi^4}{4!} + ...$ $\triangleleft$ refer to MF 27

$= \cos\pi$

$= -1$

The sum to infinity of this series is -1.

28
Solution

(a) $\displaystyle\sum_{r=2}^{\infty}\frac{r^2-r-1}{r!}=\lim_{n\to\infty}\left(\frac{n^2-n-1}{n!}\right)$ and $\displaystyle\sum_{r=2}^{\infty}\frac{r-2}{(r-1)!}=\lim_{n\to\infty}\left(\frac{n-2}{(n-1)!}\right)$.

$$\text{Consider }\lim_{n\to\infty}\left[\frac{\left(\frac{n-2}{(n-1)!}\right)}{\left(\frac{n^2-n-1}{n!}\right)}\right]=\lim_{n\to\infty}\left[\frac{n-2}{(n-1)!}\times\frac{n!}{n^2-n-1}\right]$$

$$=\lim_{n\to\infty}\frac{n^2-2n}{n^2-n-1}$$

$$=\lim_{n\to\infty}\left(1-\frac{n-1}{n^2-n-1}\right)$$

$$=1$$

Since $\displaystyle\sum_{r=2}^{\infty}\frac{r^2-r-1}{r!}$ is convergent, so $\displaystyle\sum_{r=2}^{\infty}\frac{r-2}{(r-1)!}$ is convergent by the Limit Comparison Test.

(b) $r^2-r-1>r-1,\ \text{ for } r\in\mathbb{Z}^+$

$$\frac{r-1}{r!}<\frac{r^2-r-1}{r!}\quad\text{for } r\in\mathbb{Z}^+ \text{ and } r\ge 3$$

$$\sum_{r=3}^{n}\frac{r-1}{r!}<\sum_{r=3}^{n}\frac{r^2-r-1}{r!}$$

$$\frac{1}{2}+\sum_{r=3}^{n}\frac{r-1}{r!}<\frac{1}{2}+\sum_{r=3}^{n}\frac{r^2-r-1}{r!}\qquad \triangleleft \text{ add } \frac{1}{2} \text{ on both sides}$$

Observe that $\displaystyle\sum_{r=2}^{\infty}\frac{r-1}{r!}=\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\frac{4}{5!}+\dots$

$$\sum_{r=2}^{n}\frac{r-1}{r!}<\sum_{r=2}^{n}\frac{r^2-r-1}{r!}$$

Also, $\quad 1>r-1\ \text{ for } r\in\mathbb{Z}^+$

$$\frac{1}{r!}<\frac{r-1}{r!}\quad\text{for } r\in\mathbb{Z}^+ \text{ and } r\ge 3$$

$$\sum_{r=3}^{n}\frac{1}{r!}<\sum_{r=3}^{n}\frac{r-1}{r!}$$

$$\frac{1}{2!}+\sum_{r=3}^{n}\frac{1}{r!}<\frac{1}{2!}+\sum_{r=3}^{n}\frac{r-1}{r!}$$

$$\sum_{r=2}^{n}\frac{1}{r!}<\sum_{r=2}^{n}\frac{r-1}{r!}$$

$$\sum_{r=0}^{n}\frac{1}{r!}-\frac{1}{0!}-\frac{1}{1!}<\sum_{r=2}^{n}\frac{r-1}{r!}$$

$$\therefore \quad \sum_{r=0}^{n}\frac{1}{r!}-\frac{1}{0!}-\frac{1}{1!}<\sum_{r=2}^{n}\frac{r-1}{r!}<\sum_{r=2}^{n}\frac{r^2-r-1}{r!}$$

$$\therefore \quad \lim_{n\to\infty}\left(\sum_{r=0}^{n}\frac{1}{r!}-\frac{1}{0!}-\frac{1}{1!}\right)<\lim_{n\to\infty}\sum_{r=2}^{n}\frac{r-1}{r!}<\lim_{n\to\infty}\sum_{r=2}^{n}\frac{r^2-r-1}{r!}$$

$$\mathrm{e}-2<\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\frac{4}{5!}+\ldots<2 \quad \text{(Shown)}$$

Alternative Method

$$\mathrm{e}=\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\ldots \quad \triangleleft \text{ from MF 27}$$

$$\mathrm{e}-2=\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\ldots$$

$$\sum_{r=2}^{\infty}\frac{r-1}{r!}=\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\frac{4}{5!}+\ldots$$

$$\therefore \ \mathrm{e}-2<\sum_{r=2}^{\infty}\frac{r-1}{r!}$$

$$\sum_{r=2}^{\infty}\frac{r^2-r-1}{r!}=2 \ \text{ (using GC)}$$

$$\therefore \ \sum_{r=2}^{\infty}\frac{r-1}{r!}<\sum_{r=2}^{\infty}\frac{r^2-r-1}{r!}$$

$$\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\ldots<\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\frac{4}{5!}+\ldots<\sum_{r=2}^{\infty}\frac{r^2-r-1}{r!}$$

$$\mathrm{e}-2<\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\frac{4}{5!}+\ldots<2 \quad \text{(Shown)}$$

18
Solution

(a) First term of first row $=1$

First term of second row $=1+1$

First term of third row $=1+1+2$

First term of fourth row $=1+1+2+3$

From above observation

$$\begin{aligned}\text{First term of } n\text{th row} &= 1+1+2+3+......+(n-1)\\ &= 1+\left[1+2+3+......+(n-1)\right]\\ &= 1+\frac{n-1}{2}(1+(n-1))\\ &= 1+\frac{1}{2}n(n-1) \quad \text{(Shown)}\end{aligned}$$

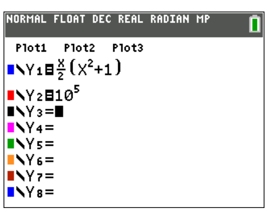
(b) Sum of the sum of all the elements nth row

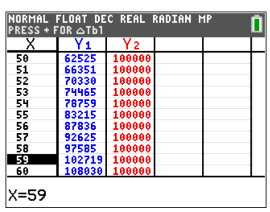
$$\begin{aligned}&= \frac{n}{2}(2a+d(n-1))\\ &= \frac{n}{2}\left(2\left(1+\frac{1}{2}n(n-1)\right)+1(n-1)\right)\\ &= \frac{1}{2}n(n^2+1)\end{aligned}$$

Given that the sum of all the integers in the nth row exceeds 10^5

i.e. $\frac{1}{2}n(n^2+1) > 10^5$

Using GC





When $n=58$, sum of elements $=97585$

When $n=59$, sum of elements $=102719$

The least value of n is 59.

19
Solution

(a) Sum of numbers in rth row

$= 1 + 2 + 3 ... + r$

$= \frac{r}{2}(1 + r)$

(b) Sum of all the numbers in nth storey

$= \sum_{r=1}^{n} \frac{r(r+1)}{2}$

$= \frac{1}{2}\sum_{r=1}^{n}(r^2 + r)$

$= \frac{1}{12}n(n+1)(2n+1) + \frac{1}{4}n(n+1)$

$= \frac{1}{12}n(n+1)(2n+1+3)$

$= \frac{1}{6}n(n+1)(n+2)$

20
Solution

(a) Time taken for robot to complete 1st stage: $3 = 2+1 = 2(1)+2^0$

Time taken for robot to complete 2nd stage: $2(2)+2 = 2(2)+2^1$

Time taken for robot to complete 3rd stage: $2(3)+2\times 2 = 2(3)+2^2$

Time taken for robot to complete 4th stage: $2(4)+2\times 3 = 2(4)+2^3$

Time taken for robot to complete Nth stage: $2N+2^{N-1}$

Total time taken

$= \sum_{r=1}^{N}\left[2r+2^{r-1}\right]$

$= \frac{(N)(2+2N)}{2}+\frac{2^N-1}{2-1}$

$= 2^N+N^2+N-1$

The total time taken for robot to complete the Nth stage is 2^N+N^2+N-1.

(b) Total distance the robot covered from its original position, to the wall and returns to its original position $= 140$ m.

Total distance travelled at the end of 1st stage $= 1$

Total distance travelled at the end of 2nd stage $= 1+2$

Total distance travelled at the end of 3rd stage $= 1+2+3$

Total distance travelled at the end of Nth stage $= 1+2+...+N$

If the robot arrives back at its original spot at the Nth stage,

i.e. $1+2+...+N \geq 140$

$N \geq 16.2$

$\therefore N = 17$

Total distance covered at the end of 16 stages

$= 1+2+3+....+16$

$= \frac{16}{2}(1\times 16)$ ◁ sum of AP

$= 136$ metres

For the robot to arrive at the original position an additional 4 metres to be completed at the 17th stage.

Time taken to complete the first 16 stages

$= (2^{16}+16^2+16-1)$ ◁ substitute $n = 16$ into 2^N+N^2+N-1

$= 65807$ seconds

Time taken for an additional 4 metres to be completed at the 17th stage

$= 2(4)$

$= 8$ seconds

Time taken for robot to arrive at original position

$=$ Time taken to complete the first 16 stages $+$ Time taken for an additional 4 metres

$= 65807+8$

$= 65815$ seconds

21
Solution

Given $u_{n+1} = u_n + 2n^2 - 5n$ .... (1)

Substitute $n = 0$ into (1)

$$u_1 = u_0$$

Substitute $n = 1$ into (1)

$$\begin{aligned} u_2 &= u_1 + 2(1)^2 - 5(1) \quad \triangleleft \text{ replace } u_1 = u_0 \\ &= u_0 - 3 \end{aligned}$$

Substitute $n = 2$ into (1)

$$\begin{aligned} u_3 &= u_2 + 2(2)^2 - 5(2) \quad \triangleleft \text{ replace } u_2 = u_1 + 2(1)^2 - 5(1) \\ &= u_1 + 2(1)^2 - 5(1) + 2(2)^2 - 5(2) \quad \triangleleft \text{ replace } u_1 = u_0 \\ &= u_0 + 2(1^2 + 2^2) - 5(1+2) \\ &= u_0 - 5 \end{aligned}$$

Substitute $n = 3$ into (1)

$$\begin{aligned} u_4 &= u_3 + 2(3)^2 - 5(3) \quad \triangleleft \text{ replace } u_3 = u_0 + 2(1^2 + 2^2) - 5(1+2) \\ &= \left[u_0 + 2(1^2 + 2^2) - 5(1+2)\right] + 2(3)^2 - 5(3) \\ &= u_0 + 2(1^2 + 2^2 + 3^2) - 5(1+2+3) \\ &= u_0 - 2 \end{aligned}$$

From the above observation,

$$\begin{aligned} \therefore \quad u_n &= u_0 + 2(1^2 + 2^2 + 3^2 + ... + (n-1)^2) - 5(1+2+3+...(n-1)) \\ &= u_0 + 2\sum_{r=1}^{n-1} r^2 - 5\sum_{r=1}^{n-1} r \\ &= u_0 + 2\left[\frac{1}{6}(n-1)n(2n-1)\right] - 5\left[\frac{(n-1)}{2}(1+n-1)\right] \\ &= u_0 + \frac{n}{3}(n-1)n(2n-1) - \frac{5n(n-1)}{2} \\ &= u_0 + \frac{n(n-1)}{6}\left[2(2n-1) - 15\right] \\ &= u_0 + \frac{1}{6}n(n-1)(4n-17) \end{aligned}$$

$\therefore \ k = \dfrac{1}{6}$

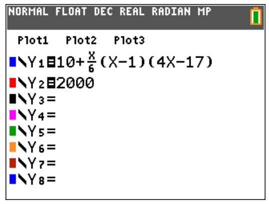
(b) Given that there are 10 cells at the start, i.e. $u_0 = 10$

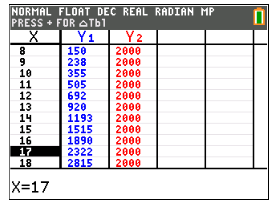
$$u_n = 10 + \frac{1}{6}n(n-1)(4n-17)$$

For the number of cells to first exceed 2000,

i.e. $$u_n > 2000$$

$$10 + \frac{1}{6}n(n-1)(4n-17) > 2000$$





From GC, $u_{16} = 1890 < 2000$

$u_{17} = 2322 > 2000$

$\therefore$ it takes 17 days for the number of bacteria to first exceed 2000.

Exercise 6

E Mixed Exercise

22
Solution

(a) $\sum_{r=1}^{\infty}\frac{1}{10^{3r}}=\frac{1}{10^3}+\frac{1}{10^6}+\frac{1}{10^9}+.....$

$$=\frac{\frac{1}{10^3}}{1-\frac{1}{10^3}}$$

$$=\frac{1}{999}$$

Express the infinite recurring decimal 0.108 in fraction gives

$$\frac{108}{1000}+\frac{108}{1000000}+\frac{108}{1000000000}+...$$

$$=108\left(\frac{1}{10^3}+\frac{1}{10^6}+\frac{1}{10^9}+...\right)$$

$$=108\left(\frac{\frac{1}{10^3}}{1-\frac{1}{10^3}}\right) \quad \triangleleft \text{ apply } S_\infty=\frac{a}{1-r}$$

$$=\frac{4}{37}$$

(b) Common ratio, $r=\frac{b}{a}$.

Given that the sum to infinity of the series is $a+2b$,

i.e. $S_\infty=a+2b$

$$\frac{a}{1-r}=a+2b \qquad \triangleleft \text{ divide } a \text{ on both sides}$$

$$\frac{1}{1-r}=1+\frac{2b}{a}$$

$$\frac{1}{1-r}=1+2r$$

$$1=1+r-2r^2$$

$$2r^2-r=0$$

$$r=0\ \left(\text{rejected } \frac{b}{a}\neq 0\right) \quad \text{or} \quad r=\frac{1}{2}$$

$\therefore$ common ratio $=\frac{1}{2}$

(b) $G_n = \dfrac{a\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}}$ ⊲ use sum of GP formulae

$$= 2a\left(1-\left(\frac{1}{2}\right)^n\right)$$

$$\sum_{n=1}^{N} G_n = 2a\sum_{n=1}^{N}\left(1-\left(\frac{1}{2}\right)^n\right) \quad \triangleleft \text{ add sum from 1 to } N$$

$$= 2a\left[N-\sum_{n=1}^{N}\left(\frac{1}{2}\right)^n\right]$$

$$= 2a\left[N-\frac{\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^N\right)}{1-\frac{1}{2}}\right]$$

$$= 2a\left(N-\left(1-\left(\frac{1}{2}\right)^N\right)\right)$$

$$= 2aN-2a\left(1-\left(\frac{1}{2}\right)^N\right) \quad \triangleleft \text{ replace } G_n = 2a\left(1-\left(\frac{1}{2}\right)^N\right) \text{ from } \textbf{(a)}$$

$$= 2aN-G_N \qquad \text{(Shown)}$$

23
Solution

(a)(i) Given $u_{n+1} = Au_n + Bn + C$

Also given $A = 1$ and $B = 0$

Substitute $A = 1$ and $B = 0$ into (1).

$\therefore \quad u_{n+1} = (1)u_n + (0)n + C$

$u_{n+1} - u_n = C,$ which implies the sequence is arithmetic progression.

$$\sum_{r=11}^{30} u_r = u_{11} + u_{11} + u_{11} + ... + u_{30}$$

$$= \frac{20}{2}(u_{11} + u_{30})$$ ◁ apply sum of AP formuale, $S_n = (a + l),$ where a is the first term and l is the last term

$$= 10(4 + 10C + 4 + 29C)$$

$$= 80 + 390C$$

Learning point:

Given that $u_1 = 4,$

Since the sequence is AP, so $u_2 = 4 + C,\ u_3 = 4 + 2C,\ u_4 = 4 + 3C$ and so on.

$\therefore\ u_{11} = 4 + 10C$ and $u_{30} = 4 + 29C$

(ii) If the sequence is a geometric progression, then recurrence relation of GP is $u_{n+1} = Au_n,$ for all positive values of n.

$\therefore\ B = 0, C = 0$

Given $\quad u_{20} > 2000$

$$u_1(A)^{19} > 2000$$

$$4(A)^{19} > 2000$$

$$A > 500^{\frac{1}{19}}$$

$\therefore\ A > 1.39$

Learning point:

Since the sequence is a geometric progression, then recurrence relation of GP is $u_{n+1} = Au_n$

Given that $u_1 = 4,$ then $u_2 = Au_1.$ < substitute $n = 1$

For $\quad u_3 = Au_2$

$u_3 = A(Au_1)$

$u_3 = A^2 u_1$

$\therefore$ for $u_{20} = A^{19} u_1$

(b) Given $u_{n+1} = Au_n + Bn + C$ .... (1)

When $u_2 = 16$, substitute $n = 1$ and $u_2 = 16$ into (1).

$u_2 = Au_1 + B(1) + C$, where $u_1 = 4$

$16 = 4A + B + C$ .... (2)

When $u_3 = 70$, substitute $n = 2$ and $u_3 = 70$ into (1).

$u_3 = Au_2 + B(2) + C$, where $u_2 = 16$

$70 = 16A + 2B + C$ .... (3)

When $u_4 = 334$, substitute $n = 2$ and $u_4 = 334$ into (1).

$u_4 = Au_3 + B(3) + C$, where $u_3 = 70$

$334 = 70A + 3B + C$ .... (4)

From GC, $A = 5, B = -6, C = 2$

$\therefore\ u_n = 5u_n - 6n + 2$ .... (5)

Substitute $n = 5$ into (5).

$u_5 = 5u_4 - 6(4) + 2$, where $u_4 = 334$

$u_5 = 5(334) - 6(4) + 2$

$\therefore\ u_5 = 1648$

24
Solution

(a) Sum of first k terms

$$=\frac{k}{2}\left[2a+(k-1)d\right]$$

Sum of the last k terms

$$= S_n - S_{n-k}$$

$$=\frac{n}{2}\left[2a+(n-1)d\right]-\frac{n-k}{2}\left[2a+(n-k-1)d\right]$$

$$=\frac{n}{2}\left[2a+(n-1)d\right]-\frac{n}{2}\left[2a+(n-k-1)d\right]+\frac{k}{2}\left[2a+(n-k-1)d\right]$$

$$=\frac{n}{2}\left[kd\right]+\frac{k}{2}\left[2a(n-k-1)d\right]$$

Difference between the sums

$$=\text{Sum of the last } k \text{ terms} - \text{Sum of the first } k \text{ terms}$$

$$=\frac{n}{2}\left[kd\right]+\frac{k}{2}\left[2a+(n-k-1)d\right]-\frac{k}{2}\left[2a+(k-1)d\right]$$

$$=\frac{n}{2}\left[kd\right]+\frac{k}{2}\left[2a+(n-k-1)d-(2a+(k-1)d\right]$$

$$=\frac{n}{2}\left[kd\right]+\frac{k}{2}\left[d(n-k-1-k+1)\right]=\frac{n}{2}\left[kd\right]+\frac{k}{2}\left[d((n-2k)\right]$$

$$= kd\left[n-k\right] \quad \text{(Shown)}$$

(b) Given $u_r = \left(\frac{1}{3}\right)^{3r-2} + \left(\frac{1}{3}\right)^{3r-1}$

$$=\left(\frac{1}{3}\right)^{3r}\left[\left(\frac{1}{3}\right)^{-2}+\left(\frac{1}{3}\right)^{-1}\right] \quad \triangleleft \text{ factor out } \left(\frac{1}{3}\right)^{3r}$$

$$=12\left(\frac{1}{3}\right)^{3r}$$

Taking sum both sides from 1 to n

$$\sum_{r=1}^{n} u_r = \sum_{r=1}^{n} 12\left(\frac{1}{3}\right)^{3r}$$

$$= 12\sum_{r=1}^{n}\left(\frac{1}{3}\right)^{3r}$$

$$= 12\left[\left(\frac{1}{3}\right)^{3} + \left(\frac{1}{3}\right)^{6} + \ldots + \left(\frac{1}{3}\right)^{3n}\right] \quad \triangleleft \text{ sum of geometric progression, where first term} = \left(\frac{1}{3}\right)^{3}, \text{ ratio} = \left(\frac{1}{3}\right)^{3} \text{ and no. of terms} = 3n$$

$$= 12\left[\frac{\left(\frac{1}{3}\right)^{3}\left(1-\left(\frac{1}{3}\right)^{3n}\right)}{1-\left(\frac{1}{3}\right)^{3}}\right]$$

$$= \frac{6}{13}\left(1-\frac{1}{27^n}\right), \text{ where } A = \frac{6}{13} \text{ and } B = 1$$

As $n \to \infty$, $\frac{1}{27^n} \to 0$.

$$\therefore \sum_{r=1}^{\infty} u_r = \frac{6}{13}$$

The sum to infinity of the series is $\frac{6}{13}$.

Alternative Method

$$u_r = \left(\frac{1}{3}\right)^{3r-2} + \left(\frac{1}{3}\right)^{3r-1}$$

$$= \left(\frac{1}{3}\right)^{3r-1}\left[\left(\frac{1}{3}\right)^{-1} + 1\right]$$

$$= 4\left(\frac{1}{3}\right)^{3r-1}$$

Taking sum both sides from 1 to n

$$\begin{aligned}\sum_{r=1}^{n} u_r &= \sum_{r=1}^{n} 4\left(\frac{1}{3}\right)^{3r-1} \\ &= 4\sum_{r=1}^{n}\left(\frac{1}{3}\right)^{3r-1} \\ &= 4\left[\left(\frac{1}{3}\right)^{2}+\left(\frac{1}{3}\right)^{5}+\ldots+\left(\frac{1}{3}\right)^{3n-1}\right] \\ &= 4\left[\frac{\left(\frac{1}{3}\right)^{2}\left(1-\left(\frac{1}{3}\right)^{3n}\right)}{1-\left(\frac{1}{3}\right)^{3}}\right] \\ &= \frac{6}{13}\left(1-\frac{1}{27^n}\right), \text{ where } A=\frac{6}{13} \text{ and } B=1\end{aligned}$$

As $n \to \infty, \dfrac{1}{27^n} \to 0.$

$\therefore \displaystyle\sum_{r=1}^{\infty} u_r = \frac{6}{13}$

The sum to infinity of the series is $\dfrac{6}{13}$.

25
Solution

(a) Given that the arithmetic sequence $u_1,\ u_2,\ u_3,\ u_4,\ ...$ has first term 2.8 and common difference 2,

$\therefore$ the general term, $u_n = 2.8 + (n-1)2$. .... (1)

Also given that the geometric sequence $v_1,\ v_2,\ v_3,\ v_4,\ ...$ has first term 3 and common ratio 1.2,

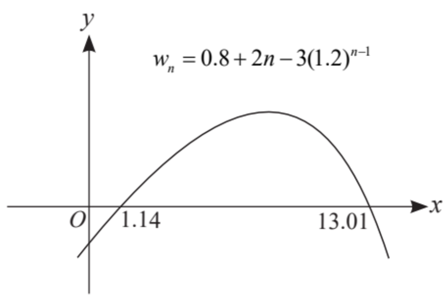
$\therefore$ the general term, $v_n = 3(1.2)^{n-1}$. .... (2)

Given $\quad w_n = u_n - v_n \qquad \triangleleft$ substitute (1) and (2)

$$= 2.8 + 2(n-1) - 3(1.2)^{n-1}$$
$$= 0.8 + 2n - 3(1.2)^{n-1}$$

$\therefore\ w_n = 0.8 + 2n - 3(1.2)^{n-1}$

(b) Use GC to graph $w_n = 0.8 + 2n - 3(1.2)^{n-1}$.



From the graph, for $w_n > 0$ when $1.14 < n < 13.01$

$\therefore$ the set of values of n is $\{n \in \mathbb{Z}^+ : 2 \le n \le 13\}$.

(c) $\sum_{n=2}^{13} w_n$

$$= \sum_{n=2}^{13} (0.8 + 2n - 3(1.2)^{n-1})$$

$$= \sum_{n=2}^{13}(0.8) + \sum_{n=2}^{13} 2n - \sum_{n=2}^{13}(3(1.2)^{n-1})$$

$$= 0.8(12) + \frac{12}{2}[4+26] - \frac{3(1.2)(1-(1.2)^{12})}{1-1.2}$$

$$= 207.6 - 18(1.2)^{12}$$

The sum of all terms w_n is $207.6 - 18(1.2)^{12}$.

26
Solution

(a) (i) $\sum_{r=0}^{n} \frac{(x+3)^r}{4^{r+1}} = \frac{1}{4}\sum_{r=0}^{n}\left(\frac{x+3}{4}\right)^r$

$$= \frac{1}{4}\left[\left(\frac{x+3}{4}\right)^0 + \left(\frac{x+3}{4}\right)^1 + \ldots + \left(\frac{x+3}{4}\right)^n\right]$$

$$= \frac{1}{4}\left[\frac{1-\left(\frac{x+3}{4}\right)^{n+1}}{1-\frac{x+3}{4}}\right]$$

$$= \frac{1}{4}\left[\frac{1-\left(\frac{x+3}{4}\right)^{n+1}}{\frac{1-x}{4}}\right]$$

$$= \frac{1}{1-x}\left[1-\left(\frac{x+3}{4}\right)^{n+1}\right]$$

(ii) $\sum_{r=0}^{n} \frac{(x+3)^r}{4^{r+1}} = \frac{1}{4}\left[\left(\frac{x+3}{4}\right)^0 + \left(\frac{x+3}{4}\right)^1 + \ldots + \left(\frac{x+3}{4}\right)^n\right]$

The above series is a geometric, with common ratio, $r = \frac{x+3}{4}$.

When $x = -5$, $r = \frac{-5+3}{4}$

$= -\frac{1}{2}$.

For geometric series to be convergence, $|r| < 1$.

Since $|r| = \frac{1}{2} < 1$, hence, the series $\sum_{r=0}^{n} \frac{(x+3)^r}{4^{r+1}}$ converges.

Take the sum from 0 to infinity,

$$\sum_{r=0}^{\infty} \frac{(-5+3)^r}{4^{r+1}} = \lim_{n\to\infty} \frac{1}{1-(-5)}\left[1-\left(\frac{-5+3}{4}\right)^{n+1}\right]$$

$$= \lim_{n\to\infty} \frac{1}{6}\left[1-\left(-\frac{1}{2}\right)^{n+1}\right]$$

$$= \frac{1}{6}$$

The limiting value is $\frac{1}{6}$.

(b) (i) Given $\sum_{r=6}^{2k} r(3r-2)$

$$= \sum_{r=6}^{2k} (3r^2 - 2r)$$

$$= \sum_{r=6}^{2k} 3r^2 - \sum_{r=6}^{2k} 2r$$

$$= 3\left[\sum_{r=1}^{2k} r^2 - \sum_{r=1}^{5} r^2\right] - 2\sum_{r=6}^{2k} r \qquad \triangleleft \text{ use } \sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$$

$$= 3\left[\frac{2k}{6}(2k+1)(4k+1) - \frac{5}{6}(6)(11)\right] - 2\left(\frac{2k-6+1}{2}(6+2k)\right)$$

$$= k(2k+1)(4k+1) - 165 - 2(2k-5)(3+k)$$

$$= k(2k+1)(4k+1) - 2(2k-5)(k+3) - 165$$

(ii) $$\sum_{r=10}^{r=66} (r-4)(3r-14) = \sum_{r+4=10}^{r+4=66} \big((r+4)-4\big)\big(3(r+4)-14\big) \qquad \triangleleft \text{ replace } r \text{ with } r+4$$

$$= \sum_{r=6}^{r=62} r(3r-2)$$

From **(b)(i),** $\sum_{r=6}^{2k} r(3r-2) = k(2k+1)(4k+1) - 2(2k-5)(k+3) - 165$

When $k = 31$, $\sum_{r=10}^{66} (r-4)(3r-14)$

$$= \sum_{r=6}^{r=62} r(3r-2)$$

$$= 31(2(31)+1)(4(31)+1) - 2(2(31)-5)((31)+3) - 165$$

$$= 240084$$

$$\therefore \sum_{r=10}^{66} (r-4)(3r-14) = 240084$$

27
Solution

(a) $\sum_{r=2}^{n} \ln\left(\frac{(r-1)e^{r^3-1}}{r}\right)$

$= \ln\left(\frac{e^{2^3-1}}{2}\right) + \ln\left(\frac{2e^{3^3-1}}{3}\right) + \ln\left(\frac{3e^{4^3-1}}{4}\right) + \ldots + \ln\left(\frac{(n-2)e^{(n-1)^3-1})}{n-1}\right) + \ln\left(\frac{(n-1)e^{n^3-1}}{n}\right)$

$= \ln\left[\left(\frac{e^{2^3-1}}{2}\right) \times \left(\frac{2e^{3^3-1}}{3}\right) \times \left(\frac{3e^{4^3-1}}{4}\right) \times \ldots \times \left(\frac{(n-2)e^{(n-1)^3-1}}{n-1}\right) \times \left(\frac{(n-1)e^{n^3-1}}{n}\right)\right]$

$= \ln\left[e^{2^3-1} \times e^{3^3-1} \times e^{4^3-1} \times \ldots \times e^{(n-1)^3-1} \times \frac{e^{n^3-1}}{n}\right]$

$= \ln e^{(2^3-1)+(3^3-1)+(4^3-1)+\ldots+(n^3-1)} - \ln n$

$= (2^3-1) + (3^3-1) + (4^3-1) + \ldots + (n^3-1) - \ln n$

$= \sum_{r=2}^{n} (r^3-1) - \ln n$

$= \sum_{r=2}^{n} r^3 + \sum_{r=2}^{n} (-1) - \ln n$

$= \sum_{r=1}^{n} r^3 - 1 - \sum_{r=2}^{n} (1) - \ln n$

$= \left(\frac{1}{2}n(n+1)\right)^2 - 1 - (n-2+1) - \ln n$

$= -\ln n + \frac{1}{4}n^2(n+1)^2 - n$

(b) $v_n = k^{\ln(u_n)-n^3}$

$w_n = v_2 \times v_3 \times v_4 \times \ldots \times v_n$

$= k^{\ln(u_2)-2^3} \times k^{\ln(u_3)-3^3} \times k^{\ln(u_4)-4^3} \times \ldots \times k^{\ln(u_n)-n^3}$

$= k^{\ln(u_2)+\ln(u_3)+\ldots+\ln(u_n)-(2^3+3^3+\ldots+n^3)}$

$= k^{\sum_{r=2}^{n}\ln(u_r) - \sum_{r=2}^{n} r^3}$

$= k^{-\ln n + \frac{1}{4}n^2(n+1)^2 - n - \left(\left(\frac{1}{2}n(n+1)\right)^2 - 1\right)}$

$= k^{-\ln n + \frac{1}{4}n^2(n+1)^2 - n - \frac{1}{4}n^2(n+1)^2 + 1}$

$= k^{-\ln n - n + 1}$

(c) $w_n = k^{-\ln n - n + 1}$

$$= \frac{k}{k^{\ln n + n}}$$

As $n \to \infty, \dfrac{1}{k^{\ln n + n}} \to 0$, thus $w_n \to 0$.

Therefore the sequence w_n is convergent.

28
Solution

(a) Given that the sum of the first three terms of the sequence is 17,

i.e. $u_1 + u_2 + u_3 = 17$ .... (1)

Also given $\sum_{r=1}^{3} S_n = 30$,

i.e. $S_1 + S_2 + S_3 = 30$

$u_1 + (u_1 + u_2) + (u_1 + u_2 + u_3) = 30$

$3u_1 + 2u_2 + u_3 = 30$ .... (2)

The third term of the sequence is twice the first term of the sequence.

i.e. $u_3 = 2u_1$

$2u_1 - u_3 = 0$........(3)

Use GC to solve (1), (2) and (3).

The first 3 terms of sequence are $u_1 = 4, u_2 = 5$ and $u_3 = 8$.

(b) After subtracting k, the three terms are consecutive terms of a GP

i.e. $r = \frac{8-k}{5-k} = \frac{5-k}{4-k}$

$32 - 12k + k^2 = 25 - 10k + k^2$

$k = 3.5$

Substitute $k = 3.5$ into $r = \frac{8-k}{5-k}$.

$\therefore\ r = \frac{8-3.5}{5-3.5} = 3$ (since $r > 1$).

The value of $k = 3.5$ and the common ratio $= 3$.

29
Solution

(a) Given $A_1 = 400$

Area of $\Delta OPR = 400\left(\frac{1}{4}\right) = 100$

$A_2 = 2 \times \text{area of } \Delta OPR = 2 \times 400\left(\frac{1}{4}\right) = 200$

Area of $\Delta STR = 200\left(\frac{1}{4}\right) = 50$

$A_3 = 2 \times \text{Area of } \Delta STR = 2 \times 50 = 100$

Hence $A_1,\ A_2,\ A_3, \ldots\ldots = 400,\ 200,\ 100,\ \ldots\ldots$

$\frac{A_2}{A_1} = \frac{A_3}{A_2} = \ldots = \frac{1}{2}$

The sequence $A_1,\ A_2,\ A_3, \ldots\ldots$ is a geometric sequence.

$A_n = 400\left(\frac{1}{2}\right)^{n-1}$, where $k = \frac{1}{2}$ $\triangleleft$ use general term of geometric sequence, where first term = 400 and ratio = 0.5

(b) $\ln A_1 + \ln A_2 + \ln A_3 + \ldots\ldots + \ln A_N$

$= \sum_{n=1}^{N} \ln A_n$

$= \sum_{n=1}^{N} \ln(400\left(\frac{1}{2}\right)^{n-1})$

$= \sum_{n=1}^{N} \left[\ln 400 + (n-1)\ln\left(\frac{1}{2}\right) \right]$

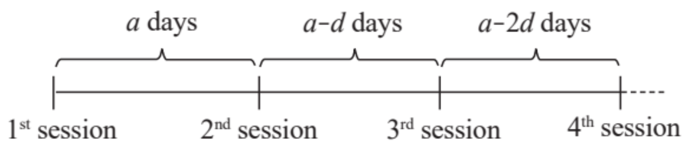
$= N\ln 400 + \ln\left(\frac{1}{2}\right) \sum_{n=1}^{N} (n-1)$

$= N\ln 400 + \ln\left(\frac{1}{2}\right)\left(\frac{N}{2}(0 + N - 1)\right)$

$= N\ln 400 + \frac{N(N-1)}{2}\ln\left(\frac{1}{2}\right)$, where $A = N$ and $B = \frac{N(N-1)}{2}$

30
Solution

(a)



Let S_n be the total number of days to complete the 9 shooting sessions.
The 1st shooting session is already attempted on the Day 1, therefore the remaining 8 shooting session must be completed less than 90 days,

i.e. $S_8 < 90$ ◁ total number of days to complete 9 shooting sessions

$$\frac{8}{2}(2a+(8-1)(-d))<90$$

$$8a-28d<90$$

$$a<11.5+3.5d \text{ .... (1)}$$

Let T_n be the duration (days) between consecutive two shooting sessions.

$$T_8>0$$

$$a+(8-1)(-d)>0$$

$$a>7d \text{ .... (2)}$$

For the last shooting session to be close to the completion date as possible, a and S_8 must be as large as possible.
By trial and error,
If $d=1$,
From (1): $a<14.75$. $\therefore$ $a=14$.
From (2): $a>7$ and $S_8=84$.
If $d=2$,
From (1): $a<18.25$. $\therefore$ $a=18$.
From (2): $a>14$ and $S_8=88$.

Therefore $d=2$, $a=18$.

(b) 92 is the highest (theoretical) point that he will get even if he practise many many times.

(c) $m = \frac{1}{9}\sum_{n=1}^{9} u_n$

$$= \frac{1}{9}\sum_{n=1}^{9}(92 - 65(b^n))$$

$$= \frac{1}{9}\left(92(9) - 65\sum_{n=1}^{9} b^n\right)$$

$$= 92 - \frac{65}{9}(b^1 + b^2 + b^3 + ... + b^9)$$

$$= 92 - \frac{65}{9}\left(\frac{b(1-b^9)}{1-b}\right)$$

The average mark for the nine training sessions is $m = 92 - \frac{65}{9}\left(\frac{b(1-b^9)}{1-b}\right)$.

(d) As he scored higher than m from his 4th session onwards,
average point for the 9 sessions $< u_4$

$$92 - \frac{65}{9}\left(\frac{b(1-b^9)}{1-b}\right) < 92 - 65(b^4)$$

$$1 - b^9 > 9b^3(1-b)$$

$$b^9 - 9b^4 + 6b^3 - 1 < 0$$

From GC, $0 < b < 0.726$

$\therefore$ the range of values of b is $0 < b < 0.726$.

31
Solution

(a) Number of seats form an AP with 1st term 21, common difference 2 and number of rows = 20.

Total number of seats installed in the concert hall

$=\frac{20}{2}[2(21)+(20-1)(2)]$

$=800$

There are 800 seats to be installed in the hall.

(b) From **(a)**, there are 800 seats to be installed in the concert hall.

Given that each order allows multiples of 50 seats,

$\therefore$ $800\div 50=16$ sets of 50 seats can be ordered.

Cost of one seat in the set of 50 seats from Supplier A

Cost of one seat in the 1st set of 50 seats = 90

Cost of one seat in the 2nd set of 50 seats = 0.9(90)

Cost of one seat in the 3rd set of 50 seats = $0.9^2(90)$

$\therefore$ cost of one seat in the nth set of 50 seats = $0.9^{n-1}(90)$ ◁ GP progression

$$90(0.9)^{n-1}\ge 40$$
$$n-1\le 7.6967$$
$$n\le 8.6967$$

$\therefore$ $n=1,\ 2,\ ...,\ 8$

Total cost of ordering the seats from Supplier A

= (Number of seats) × (Cost of one seat in the 1st set of 50 seats + Cost of one seat in the 2nd set of 50 seats + ... + Cost of one seat in the 8th set of 50 seats + Cost of remaining 8 rows at \$40 per seat)

$=\$50[90+90(0.9)+90(0.9)^2+...+90(0.9)^7+40(8)]$

$=\$4500\left[\frac{1-0.9^8}{1-0.9}\right]+16000$

= \$41628.98 (nearest cent)

Cost of one seat in the set of 50 seats from Supplier B

Cost of one seat in the 1st set of 50 seats = 90

Cost of one seat in the 2nd set of 50 seats = $90+(-7)$

Cost of one seat in the 3rd set of 50 seats = $90+2(-7)$

Cost of one seat in the nth set of 50 seats = $90+(n-1)(-7)$ ◁ AP progression

$$90+(n-1)(-7)\ge 40$$
$$n\le 8$$

$\therefore$ $n=1,\ 2,\ ...,\ 8$

Total cost of ordering the seats from Supplier B

$= (\text{Number of seats}) \times (\text{Cost of one seat in the 1st set of 50 seats} + \text{Cost of one seat in the 2nd set of 50 seats} + \ldots$

$+ \text{Cost of one seat in the 8th set of 50 seats} + \text{Cost of remaining 8 rows at \$40 per seat})$

$= \$50\left[\frac{8}{2}[2(90)+(8-1)(-7)]+40(8)\right]$

$= \$42200$

Since $41628.98 < 42200$, the management should order the chairs from Supplier A.

(c) Total value of the discount coupons issued to the first $(100+n)$ audiences

=Discount for first 100 audiences + Discount for n audiences

$= 10(100)+0.1(1)^2+0.1(2)^2+\ldots\ldots+0.1(n)^2$

$= 1000+0.1\left[(1)^2+(2)^2+\ldots\ldots+(n)^2\right]$

$= 1000+0.1\sum_{r=1}^{n} r^2$

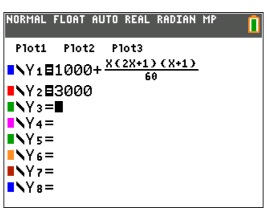
$= 1000+0.1\left[\frac{n(2n+1)(n+1)}{6}\right]$

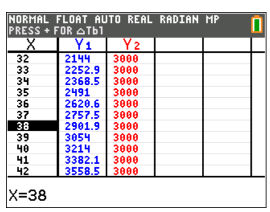
$= 1000+\frac{n(2n+1)(n+1)}{60}$

(d) Given that the management sets aside a total of \$3000 for the discount coupons,

i.e. total value of the discount coupons ≤ 3000

$$1000+\frac{n(2n+1)(n+1)}{60} \leq 3000$$





From GC, the maximum value of $n = 38$

The maximum number of audiences who will receive a coupon is 38.

Exercise 6

F Higher Order Questions

32
Solution

Regrouping the series $2+3+4+6+8+9+16+12+32+15+64+18+128+21+256+...$

$= (2+4+8+16+...n\text{th term}) + (3+6+9+...n\text{th term})$

$= (2+4+8+16+...+2^n) + (3+6+9+... +2n)$

$= \sum_{r=1}^{n} 2^r + \sum_{r=1}^{n} 3r$

$= \sum_{r=1}^{n} (2^r + 3r)$

$\therefore \quad a = 3$ and $b = 2$

$\sum_{r=1}^{2n} 3r + 2^r$

$= 3\sum_{r=1}^{n} r + \sum_{r=1}^{n} 2^r$

$= 3\left[\frac{2n(2n+1)}{2}\right] + \left[\frac{2^n(2^{2n}-1)}{1}\right]$

$= 3n(2n+1) + 2(2^{2n}-1)$

33
Solution

(a) Given $n < M+1$, where n and M are positive integers

$\therefore\ 0 < M+1-n$

If $n=1$, $u_1 = \dfrac{3}{M}$

If $n=2$, $u_2 = \dfrac{3}{M-1}$

If $n=3$, $u_3 = \dfrac{3}{M-2}$

When $n \to \infty$, $\dfrac{3}{M-n+1} \to 3$

The sequence is increasing from $\dfrac{3}{M}$ to 3.

(b) $S_1 = u_1 = \dfrac{3}{M-1+1} = \dfrac{3}{M}$

$S_2 = u_1 + u_2 = \dfrac{3}{M} + \dfrac{3}{M-2+1} = \dfrac{3}{M} + \dfrac{3}{M-1}$

$S_3 = u_1 + u_2 + u_3 = \dfrac{3}{M} + \dfrac{3}{M-1} + \dfrac{3}{M-2}$

$$\sum_{n=1}^{M} S_n$$
$$= S_1 + S_2 + S_3 + \ldots + S_M$$
$$= (u_1) + (u_1 + u_2) + (u_1 + u_2 + u_3) + \ldots + (u_1 + \ldots + u_M)$$
$$= Mu_1 + (M-1)u_2 + (M-2)u_3 + \ldots + u_M$$
$$= M\left(\frac{3}{M}\right) + (M-1)\left(\frac{3}{M-1}\right) + (M-2)\left(\frac{3}{M-2}\right) + \ldots + 3$$
$$= 3 + 3 + 3 + \ldots. + 3$$
$$= 3M$$

(c) $S_n = u_1 + u_2 + \ldots + u_n$

$$= \frac{3}{M} + \frac{3}{M-1} + \ldots + \frac{3}{M-n+1}$$

Sum of $\dfrac{3}{M} + \dfrac{3}{M} + \ldots + \dfrac{3}{M} + \ldots\ldots$ (n terms) $= \dfrac{3n}{M}$

$$\therefore\ \frac{3}{M} + \frac{3}{M-1} + \ldots + \frac{3}{M-n+1} > \frac{3}{M} + \frac{3}{M} + \ldots + \frac{3}{M} + \ldots\ldots\ (n \text{ terms})$$

$$S_n > \frac{3n}{M} \qquad \text{(Shown)}$$

34
Solution

(a) Given $a_{n+1} = a_n + ka_{n-1}$

When $n = 1,\ a_2 = a_1 + ka_0$ .... (1)

Substitute $a_0 = 2, a_1 = 7$ and $a_2 = 11$ into (1)

$$11 = 7 + k(2)$$
$$k = 2$$

$\therefore\ k = 2$

(b) Given $a_n = A(2^n) + B(-1)^n + C$ .... (2)

Substitute $n = 0$ into (2)

$$a_0 = A(2^0) + B(-1)^0 + C \qquad \triangleleft\ a_0 = 2 \text{ from } \textbf{(a)}$$
$$2 = A + B + C \ \text{..................... (3)}$$

Substitute $n = 1$ into (2)

$$a_1 = A(2^1) + B(-1)^1 + C \qquad \triangleleft\ a_1 = 7 \text{ from } \textbf{(a)}$$
$$7 = 2A - B + C \ \text{................... (4)}$$

Substitute $n = 2$ into (2)

$$a_2 = A(2^2) + B(-1)^2 + C \qquad \triangleleft\ a_2 = 11 \text{ from } \textbf{(a)}$$
$$11 = 4A + B + C \ \text{................... (5)}$$

Using GC to solve (3), (4) and (5).

$\therefore\ A = 3, B = -1, C = 0$

(c) $a_1 + a_2 + a_3 + ... + a_n$

$$= \sum_{r=1}^{n} a_r$$

$$= \sum_{r=1}^{n} \left[3(2^r) - (-1)^r \right]$$

$$= 3\sum_{r=1}^{n} (2^r) - \sum_{r=1}^{n} (-1)^r$$

When n is odd,

$$= 3\left[\frac{2(2^n - 1)}{2 - 1} \right] - (-1)$$

$$= 6(2^n - 1) + 1$$

$$= 6(2^n) - 5$$

$$\sum_{r=1}^{n} a_r$$

$$= \sum_{r=1}^{n}\left[3(2^r)-1(-1)^r\right]$$

$$= 3\sum_{r=1}^{n}(2^r)-\sum_{r=1}^{n}(-1)^r$$

When n is even,

$$= 3\left[\frac{2(2^n-1)}{2-1}\right]$$

$$= 6(2^n-1)$$

35
Solution

(a) Given $u_{n+1} = 0.1u_n + k$ .... (1)

Substituting $n = 1$ and $u_1 = -3$ into (1)

$$u_2 = 0.1(u_1) + k$$
$$= 0.1(-3) + k$$
$$= k - 0.3$$

Substituting $n = 2$ and $u_2 = k - 0.3$ into (1)

$$u_3 = 0.1u_2 + k$$
$$= 0.1(k - 0.3) + k$$
$$= 1.1k - 0.03$$

(b) Given $u_n = l + m\left(\frac{1}{10}\right)^n$ .... (2)

Substituting $n = 1$ into (2)

$$-3 = l + m\left(\frac{1}{10}\right)^1$$
$$-3 = l + 0.1m \text{ .... (3)}$$

Substituting $n = 2$ into (2)

$$k - 0.3 = l + m\left(\frac{1}{10}\right)^2$$
$$k - 0.3 = l + 0.01m \text{ .... (4)}$$

Take $(3) - (4)$:

$$-3 - (k - 0.3) = (l + 0.1m) - (l + 0.01m)$$
$$-2.7 - k = 0.09m$$
$$m = \frac{100}{9}(-2.7 - k)$$
$$= -\frac{100}{9}k - 30$$

$\therefore\ m = -\frac{100}{9}k - 30$ .... (5) (Shown)

Substituting $m = -\frac{100}{9}k - 30$ into (1)

$$-3 = \beta + \frac{1}{10}\left(-\frac{100}{9}k - 30\right)$$
$$-3 = \beta - \frac{10}{9}k - 3$$
$$\beta = \frac{10}{9}k$$

$\therefore\ \beta = \frac{10}{9}k$ .... (6)

(c) $\lim_{n\to\infty} u_n$

$$= \lim_{n\to\infty}\left(\frac{10}{9}k + m\left(\frac{1}{10}\right)^n\right)$$

$$= \frac{10}{9}k$$

When n becomes large, the value of u_n is $\frac{10}{9}k$.

(d) Given that $k = 9$, substituting $k = 9$ into (5) and (6)

From (5): $m = -\frac{100}{9}(9) - 30$

$= -130$

From (6): $l = \frac{10}{9}(9)$

$l = 10$

$\therefore\ u_n = 10 - 130\left(\frac{1}{10}\right)^n$

$$\sum_{n=1}^{N} u_n$$

$$= \sum_{n=1}^{N}\left[10 - 130\left(\frac{1}{10}\right)^n\right]$$

$$= \sum_{r=1}^{N} 10 - 130\sum_{i=1}^{n}\left(\frac{1}{10}\right)^n$$

$$= 10N - (130)\frac{\frac{1}{10}\left(1 - \frac{1}{10^N}\right)}{1 - \frac{1}{10}}$$

$$= 10N - \frac{130}{9}\left(1 - \frac{1}{10^N}\right)$$

$$= 10N - \frac{130}{9} + \frac{130}{9(10^N)}$$

$$\therefore\ \sum_{n=1}^{N} u_n = 10N - \frac{130}{9} + \frac{130}{9(10^N)}$$

36
Solution

(a) From $A_k = A_{k-1} + 360k^2$ .... (1)

Substitute $k = 1$ into (1)

$$A_1 = A_0 + 360(1)^2$$
$$= A_0 + 360$$

Substitute $k = 2$ into (1)

$$A_2 = A_1 + 360(2)^2 \qquad \triangleleft \text{ replace } A_1 = A_0 + 360$$
$$= \left[A_0 + 360(1)^2 \right] + 360(2)^2$$
$$= A_0 + 1800$$

Substitute $k = 3$ into (1)

$$A_3 = A_2 + 360(3)^2 \qquad \triangleleft \text{ replace } A_2 = A_0 + 360(1)^2 + 360(2)^2$$
$$= \left[A_0 + 360(1)^2 + 360(2)^2 \right] + 360(3)^2$$
$$= A_0 + 360(1)^2 + 360(2)^2 + 360(3)^2$$
$$= A_0 + 5040$$

From the above observation,

$$A_n = A_0 + 360(1)^2 + 360(2)^2 + 360(3)^2 + \ldots + 360(n)^2$$
$$= A_0 + 360\left[(1)^2 + (2)^2 + (3)^3 + \ldots + (n)^2 \right]$$
$$= A_0 + 360\left[\sum_{r=1}^{n} r^2 \right]$$
$$= A_0 + 360\left[\frac{n}{6}(n+1)(2n+1) \right]$$
$$= A_0 + 60n\left[2n^2 + 3n + 1 \right]$$
$$= A_0 + 120n^3 + 180n^2 + 60n$$

$\therefore\ a = 120,\ b = 180,\ c = 60$

(b) $\mathrm{P}(n)-\mathrm{P}(n-1)<0$

$$\sum_{r=1}^{2n}(450-nr)-\sum_{r=1}^{2(n-1)}\left[450-(n-1)r\right]<0$$

$$\sum_{r=1}^{2n}450-n\sum_{r=1}^{2n}r-\sum_{r=1}^{2n-2}450+(n-1)\sum_{r=1}^{2n-2}r<0$$

$$\sum_{r=1}^{2n}450-\sum_{r=1}^{2n-2}450-n\sum_{r=1}^{2n}r+(n-1)\sum_{r=1}^{2n-2}r<0$$

$$2(450)-n\left[\frac{2n}{2}(1+2n)\right]+(n+1)\left[\frac{2n-2}{2}(1+2n-2)\right]<0$$

$$900-n^2(1+2n)+(n-1)^2(2n-1)<0$$

$$899-6n^2+4n<0$$

Using G.C,

$n>12.6$ or $n<-11.9$ (rejected since $n>0$)

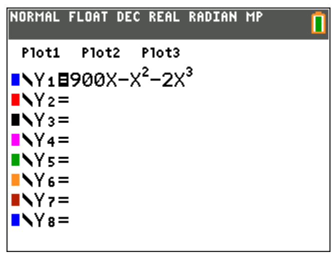
Hence, it will take 13 days for the level of bacteria to start to decline after the antibody is administered.

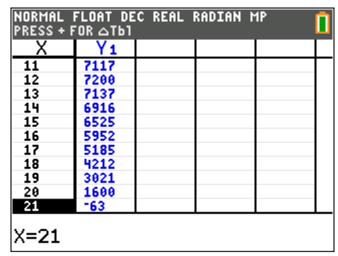
(c) Let $\mathrm{P}(n)=0$

$$\sum_{r=1}^{2n}(450-nr)=0$$

$$900n-n^2-2n^3=0$$

Using G.C to solve $900n-n^2-2n^3=0$.





It will need a total of 21 days for the bacteria to be entirely cleared from the body.

(d) $\mathrm{Q}(n)=1617-20(n-7)^2$

Using G.C,

$n=6, \mathrm{Q}(6)=1597$

$n=7, \mathrm{Q}(7)=1617$

$n=8, \mathrm{Q}(8)=1597$

$\mathrm{P}(n)=900n-n^2-2n^3$

Using G.C,

$n=11, \mathrm{P}(11)=7117$

$n=12, \mathrm{P}(12)=7200$

$n=13, \mathrm{P}(13)=7137$

It takes 8 days for the amount of bacteria present to be reduced when using the oral medication whereas 13 days are required when using the antibody.
Thus, the oral medication is more effective in reducing the amount of bacteria present.

37
Solution

(a) Let u_n denote the number of families evacuating the village at the end of the nth day after the storm arrived.

$$\begin{aligned} u_n &= 0 \\ 40+(n-1)(-2) &= 0 \\ 42-2n &= 0 \\ n &= 21 \end{aligned}$$

Hence the last family evacuated the village at the end of the 20th day after the storm arrived.

Learning point:
$n = 21$ means no more families are evacuated at the end of the 21st day after the storm struck.

(b) Let S_n denote the total number of families that have evacuated the village by the end of the nth day.

Total number of families that evacuated the village
$= S_{20}$
$= \dfrac{20}{2}(40+2)$
$= 420$

Total number of families remaining after all evacuation is complete
$= 600 - 420$
$= 180$

There are 180 number of families remain in the village after no more families evacuate the village

(c) Let v_n denote the number of families remaining in the village at the end of the nth day after the storm struck.
$v_n = 600 - S_n$
$= 600 - \dfrac{n}{2}[2(40)+(n-1)(-2)]$
$= 600 - 41n + n^2$ (Shown)

(d) Number of units supplied in the first 20 days
$= \displaystyle\sum_{r=1}^{20}(600-41r+r^2)$
$= \displaystyle\sum_{r=1}^{20}600 + \sum_{r=1}^{20}41r + r^2 + \sum_{r=1}^{20}r^2$
$= 600(20) - 41\left[\dfrac{20}{2}(1+20)\right] + \dfrac{20}{6}(20+1)(2(20)+1)$
$= 6260$

Remaining supplies can last

$$= \frac{11000 - 6260}{180 \text{ families}}$$

$$= 26\frac{1}{3} \text{ days.}$$

Number of days that the storage facility is unable to supply the village's families with supplies

$$= 20 + 27$$

$$= 47 \text{ days}$$

The storage facility will be unable to supply the village's families with supplies on the 47 days.

Exercise 6

G Exam Style Questions

38
Solution

(a) $S_n = \sum_{r=1}^{n}(3r+5)$

$= 3\sum_{r=1}^{n} r + \sum_{r=1}^{n} 5$

$= \frac{3n}{2}(n+1) + 5n$

$= \frac{3}{2}n^2 + \frac{13}{2}n$

(b) $u_n = \sum_{r=1}^{n} u_r - \sum_{r=1}^{n-1} u_r$

$= n^2 - 3n - \left[(n-1)^2 - 3(n-1)\right]$

$= n^2 - 3n - \left[n^2 - 2n + 1 - 3n + 3\right]$

$= 2n - 4$

The nth term of the series is $2n-4$.

39
Solution

(a) Given $u_{n+1} = 2u_n + A_n$ .... (1)

Substituting $n=1,\ u_1 = 5$ and $u_2 = 15$ into (1)

$$u_2 = 2u_1 + A(1)$$
$$15 = 2(5) + A$$
$$A = 5$$

Substituting $n=1$ and $u_2 = 15$ into (1)

$$u_3 = 2u_2 + 5(2)$$
$$= 2(15) + 5(2)$$
$$= 40$$

$\therefore\ A = 5$ and $u_3 = 40$

(b) Let $u_n = a(2^n) + bn + c$ .... (2)

Substituting $n=1$ and $u_1 = 5$ into (2)

$$u_1 = a(2^1) + b(1) + c$$
$$5 = 2a + b + c \text{ .... (3)}$$

Substituting $n=2$ and $u_2 = 15$ into (2)

$$u_2 = a(2^2) + b(2) + c$$
$$15 = 4a + 2b + c \text{ .... (4)}$$

Substituting $n=2$ and $u_3 = 40$ into (2)

$$u_3 = a(2^3) + b(3) + c$$
$$40 = 4a + 2b + c$$
$$40 = 8a + 3b + c \text{ .... (5)}$$

Use GC to solve (3), (4) and (5).

$\therefore\ a = 7.5,\ b = -5,\ c = -5$

(c) Substituting $a = 7.5,\ b = -5$ and $c = -5$ into (2)

$\therefore\ u_n = 7.5(2^n) - 5n - 5$

$$\sum_{r=1}^{n}(7.5(2^r) - 5r - 5)$$

$$= 7.5\sum_{r=1}^{n}2^r - 5\sum_{r=1}^{n}r - \sum_{r=1}^{n}5$$

$$= 7.5\left(\frac{2(2^n-1)}{2-1}\right) - 5\left(\frac{n}{2}(1+n)\right) - 5n$$

$$= 15(2^n-1) - 5n\left(\frac{1}{2}(1+n)+1\right)$$

$$= 15(2^n-1) - \frac{5n}{2}(3+n)$$

40
Solution

(a) $\sum_{r=1}^{n}(2r-7)(r+1)$

$=\sum_{r=1}^{n}(2r^2-5r-7)$

$=2\sum_{r=1}^{n}r^2-\sum_{r=1}^{n}(5r+7)$

$=2\left[\frac{1}{6}n(n+1)(2n+1)\right]-\frac{n}{2}\left[(5(1)+7)+(5(n)+7)\right]$

$=2\left[\frac{1}{6}n(n+1)(2n+1)\right]-\frac{n}{2}\left[(5n+19)\right]$ ◁ factor out $\frac{n}{6}$

$=\frac{1}{6}n\left[2(2n^2+3n+1)-3(5n+19)\right]$

$=\frac{1}{6}n\left[4n^2+6n+2)-(15n+57)\right]$

$=\frac{1}{6}n(4n^2-9n-55)$ (Shown)

(b) $\sum_{r=1}^{n}3^{-r}$

$=\sum_{r=1}^{n}\left(\frac{1}{3}\right)^r$

$=\left(\frac{1}{3}\right)^1+\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^3+....+\left(\frac{1}{3}\right)^n$ ◁ geometric series

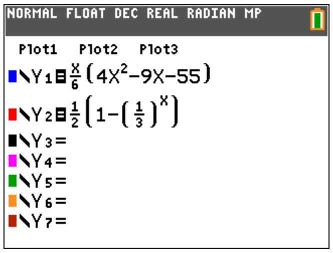
$=\dfrac{\frac{1}{3}\left[1-\left(\frac{1}{3}\right)^n\right]}{1-\frac{1}{3}}$ ◁ apply sum of first nth term of geometric series formulae, $S_n=\dfrac{a(1-r^n)}{1-r}$

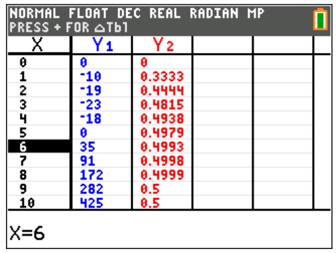
$=\frac{1}{2}\left[1-\left(\frac{1}{3}\right)^n\right]$

For $\sum_{r=1}^{n}(2r-7)(r+1) > \sum_{r=1}^{n} 3^{-r}$

$$\frac{1}{6}n(4n^2-9n-55) > \frac{1}{2}\left[1-\left(\frac{1}{3}\right)^n\right]$$

From GC,





The least value of n is 6.

(c) $\sum_{r=n+1}^{2n}(2r-7)(r+1)$

$$=\left[\frac{1}{6}(2n)(4(2n)^2-9(2n)-55)\right]-\left[\frac{1}{6}n(4n^2-9n-55)\right]$$

$$=\frac{1}{6}n\left[2(16n^2-18n-55)-(4n^2-9n-55)\right]$$

$$=\frac{1}{6}n\left[(32n^2-36n-110)-(4n^2-9n-55)\right]$$

$$=\frac{1}{6}n(28n^2-27n-55) \text{ or } \frac{14}{3}n^3-\frac{9}{2}n^2-\frac{55}{6}n$$

(d) $43\times 26+45\times 27+47\times 28+\ldots+87\times 48+89\times 49$

$$=\sum_{r=25}^{48}(2r-7)(r+1)$$

$$=\sum_{r=1}^{48}(2r-7)(r+1)-\sum_{r=1}^{24}(2r-7)(r+1)$$ ◁ use the result in **(a)**: $\sum_{r=1}^{n}(2r^2-5r-7)=\frac{1}{6}n(4n^2-9n-55)$

$$=\frac{1}{6}(48)\left[4(48)^2-9(48)-55\right]-\frac{1}{6}(24)\left[4(24)^2-9(24)-55\right]$$

$$=61700$$

41
Solution

(a) $2^3+4^3+6^3+\ldots+(2n)^3$

$=\sum_{r=1}^{n}(2r)^3$

$=8\sum_{r=1}^{n}r^3$

$=8\left[\frac{n^2(n+1)^2}{4}\right]$

$=2n^2(n+1)^2$ (Shown)

$\sum_{r=1}^{n}(2r-1)^3$

$=\sum_{r=1}^{2n}r^3-\sum_{r=1}^{n}(2r)^3$ ⊲ use the result in **(a)**: $\sum_{r=1}^{n}(2r)^3=2n^2(n+1)^2$ and $\sum_{r=1}^{n}r^3=\frac{n^2(n+1)^2}{4}$

$=\frac{(2n)^2(2n+1)^2}{4}-2n^2(n+1)^2$

$=n^2(2n+1)^2-2n^2(n+1)^2$

$=n^2(4n^2+4n+1-2n^2-4n-2)$

$=n^2(2n^2-1)$ (Shown)

(b) $\sum_{r=1}^{2n}(-1)^r(r)^3$

$=-1^3+2^3-3^3+4^3-5^3+6^3-\ldots+(2n)^3-(2n-1)^3$

$=\left[2^3+4^3+6^3+\ldots+(2n)^3\right]-1^3-3^3-5^3-\ldots-(2n-1)^3$

$=\left[2^3+4^3+6^3+\ldots+(2n)^3\right]-\left[1^3+3^3+5^3+\ldots+(2n-1)^3\right]$

$=\sum_{r=1}^{n}(2r)^3-\sum_{r=1}^{n}(2r-1)^3$ ⊲ use the result in **(a)**

$=2n^2(n+1)^2-n^2(2n^2-1)$

$=n^2\left[2(n^2+2n+1)-2n^2+1\right]$

$=n^2(4n+3)$

42
Solution

(a) Let u_n be the nth term of the sequence.
Given that S_1, S_2, S_3 and S_4 are $2, 6, 18$ and 44.

$$u_1 = S_1 = 2$$
$$u_2 = S_2 - S_1 = 6 - 2 = 4$$
$$u_3 = S_3 - S_2 = 18 - 6 = 12$$

$\therefore\ u_1 = 2,\ u_2 = 4$ and $u_3 = 12$

If S_n is a quadratic polynomial, then $S_n = an^2 + bn + c$, where a, b and c are constants.
Also the general term, $u_n = S_n - S_{n-1}$ must be a linear polynomial in terms of n.
Since $u_3 - u_2 \neq u_2 - u_1, u_n$ cannot be linear polynomial in terms of n. Therefore S_n cannot be a quadratic polynomial.

Alternative Method

Let $S_n = an^2 + bn + c$ .... (1)
Substituting $n = 1$ and $S_1 = 2$ into (1)
$a + b + c = 2.$ .... (2)
Substituting $n = 2$ and $S_2 = 6$ into (1)
$4a + 2b + c = 6$ .... (3)
Substituting $n = 3$ and $S_3 = 8$ into (1)
$9a + 3b + c = 18$ .... (4)
Substituting $n = 4$ and $S_4 = 44$ into (1)
$16a + 4b + c = 44$ .... (5)
Solving the 4 equations will result in no solutions. Therefore $S_n \neq an^2 + bn + c.$

(b) Let $S_n = an^3 + bn^2 + cn + d$ .... (1)
Given that S_1, S_2, S_3 and S_4 are $2, 6, 18$ and 44 respectively,
Substituting $n = 1$ and $S_1 = 2$ into (1)
$\therefore \quad a + b + c + d = 2$ .... (2)
Substituting $n = 2$ and $S_2 = 6$ into (1)
$8a + 4b + 2c + d = 6$ .... (3)
Substituting $n = 3$ and $S_3 = 8$ into (1)
$27a + 9b + 3c + d = 18$ .... (4)
Substituting $n = 4$ and $S_4 = 44$ into (1)
$64a + 16b + 4c + d = 44$ .... (5)
Using GC, $a = 1, b = -2, c = 3, d = 0.$

$\therefore\ S_n = n^3 - 2n^2 + 3n$

(c) $S_n - S_{n-1}$

$= n^3 - 2n^2 + 3n - (n-1)^3 + 2(n-1)^2 - 3(n-1)$

$= n^3 - 2n^2 + 3n - (n^3 - 3n^2 + 3n - 1) + 2(n^2 - 2n + 1) - 3n + 3$

$= 3n^2 - 7n + 6$

$\sum_{n=1}^{N} S_n - S_{n-1}$

$= \sum_{n=1}^{N} 3n^2 - 7n + 6$

$= 3\sum_{n=1}^{N} n^2 - 7\sum_{n=1}^{N} n + 6N$

$= 3\sum_{n=1}^{N} n^2 - \frac{7N(N+1)}{2} + 6N$

$= \frac{N}{6}(N+1)(2N+1) - \frac{7N(N+1)}{2} + 6N$

43
Solution

(a) Let the sequence $\{u_n\}$ converges be L.

Then $L = \dfrac{2}{2-L}$

$$2L - L^2 = 2$$

$$L^2 - 2L + 2 = 0 \quad \triangleleft \text{ completing the square}$$

$$(L-1)^2 + 1 = 0$$

Since there is no real solution for L, the sequence does not converge.

(b)(i) Using GC

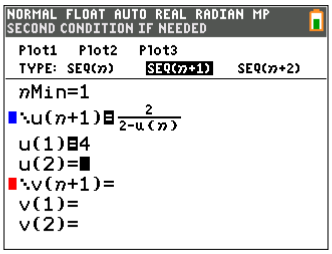
$u_1 = 4$

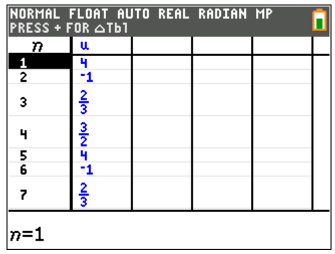
$u_2 = -1$

$u_3 = \dfrac{2}{3}$

$u_4 = \dfrac{3}{2}$

$u_5 = 4$





The next 4 terms of the sequence are $4, -1, \dfrac{2}{3}, 4$

(b)(ii) $u_{n+1} = \dfrac{2}{2-u_n}$ .... (1)

Replace n by $n+1$ in (1)

$u_{n+2} = \dfrac{2}{2-u_{n+1}}$ .... (2)

Substitute (1) into (2)

$$u_{n+2} = \frac{2}{2 - \left(\dfrac{2}{2-u_n}\right)}$$

$u_{n+2} = \dfrac{2(2-u_n)}{2-2u_n}$ .... (3)

$u_{n+2} = \dfrac{2-u_n}{1-u_n}$ .... (4)

Replace n by $n+2$ in (4)

$u_{n+4} = \dfrac{2-u_{n+2}}{1-u_{n+2}}$ .... (5)

Substitute (4) into (5)

$$u_{n+4} = \frac{2 - \left(\dfrac{2-u_n}{1-u_n}\right)}{1 - \left(\dfrac{2-u_n}{1-u_n}\right)}$$

$$u_{n+4} = \frac{\dfrac{2(1-u_n)-(2-u_n)}{1-u_n}}{\dfrac{(1-u_n)-(2-u_n)}{1-u_n}}$$

$$u_{n+4} = \frac{2-2u_n-2+u_n}{1-u_n-2+u_n}$$

$$= \frac{-u_n}{-1}$$

$\therefore \quad u_{n+4} = u_n \quad$ (Shown)

44
Solution

(a)

Given $u_n = u_{n-1} + n + 2^n$ .... (1)

Replace n by $n-1$ in (1)

$$u_{n-1} = u_{n-2} + (n-1) + 2^{n-1} \text{ .... (2)}$$

Replace n by $n-2$ in (1)

$$u_{n-2} = u_{n-3} + (n-2) + 2^{n-2} \text{ .... (3)}$$

Replace n by $n-3$ in (1)

$$u_{n-3} = u_{n-4} + (n-3) + 2^{n-3}$$

From (1): $u_n = u_{n-1} + n + 2^n$ ◃ replace u_{n-1} by (2)

$= u_{n-2} + (n-1) + 2^{n-1} + n + 2^n$ ◃ replace u_{n-2} by (3)

$= u_{n-3} + (n-2) + 2^{n-2} + (n-1) + n + 2^{n-1} + 2^n$ ◃ arrange in sequence

$= u_{n-3} + (n-2) + (n-1) + n + 2^{n-2} + 2^{n-1} + 2^n$

From observation,

$= u_1 + \left[2+3+...+(n-2)+(n-1)+n\right] + \left[2^2 + 2^3 + ... + 2^{n-2} + 2^{n-1} + 2^n\right]$ ◃ given that $u_1 = 4$

$= 4 + \left[2+3+...+(n-2)+(n-1)+n\right] + \left[2^2 + 2^3 + ... + 2^{n-2} + 2^{n-1} + 2^n\right]$

$$= 4 + \frac{(n-1)(n+2)}{2} + \frac{2^2(2^{n-1}-1)}{2-1}$$

$$= 4 + \frac{(n-1)(n+2)}{2} + 4(2^{n-1}) - 4$$

$$= \frac{(n-1)(n+2)}{2} + 2^{n+1}$$

Learning point:

Use sum of AP formulae to find $\left[2+3+...+(n-2)+(n-1)+n\right]$

Use sum of GP formulae to find $\left[2^2 + 2^3 + ... + 2^{n-2} + 2^{n-1} + 2^n\right]$

(b) For $n \ge 2$, $u_n > u_{n-1} > u_{n-2} > ... > u_1$ and as $n \to \infty$, $\dfrac{(n-1)(n+2)}{2} \to \infty$ and $2^{n+1} \to \infty$

$\therefore\ u_n \to \infty$

Hence the sequence is increasing and divergent.

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Solution

(a) Given $4T_{n+1} = aT_n + 2$ and $T_1 = 20$

When $n = 1$,

$$T_2 = \frac{a}{4}U_1 + \frac{1}{2}$$

$$T_2 = \frac{a}{4}(20) + \frac{1}{2}$$

When $n = 2$,

$$U_3 = \frac{a}{4}U_2 + \frac{1}{2}$$

$$U_3 = \frac{a}{4}\left(\frac{a}{4}(20) + \frac{1}{2}\right) + \frac{1}{2}$$

$$= \left(\frac{a}{4}\right)^2 (20) + \frac{a}{4}\left(\frac{1}{2}\right) + \frac{1}{2}$$

When $n = 3$,

$$U_4 = \frac{a}{4}U_3 + \frac{1}{2}$$

$$U_4 = \frac{a}{4}\left(\left(\frac{a}{4}\right)^2 (20) + \frac{a}{4}\left(\frac{1}{2}\right) + \frac{1}{2}\right) + \frac{1}{2}$$

$$= \left(\frac{a}{4}\right)^3 (20) + \left(\frac{a}{4}\right)^2 \left(\frac{1}{2}\right) + \left(\frac{a}{4}\right)\left(\frac{1}{2}\right) + \frac{1}{2}$$

(b) $U_n = \left(\frac{a}{4}\right)^{n-1} (20) + \left(\frac{a}{4}\right)^{n-2} \left(\frac{1}{2}\right) + \left(\frac{a}{4}\right)^{n-3} \left(\frac{1}{2}\right) + \ldots\ldots \left(\frac{a}{4}\right)^2 \left(\frac{1}{2}\right) + \left(\frac{a}{4}\right)\left(\frac{1}{2}\right) + \frac{1}{2}$

$$= \left(\frac{a}{4}\right)^{n-1} (20) + \frac{1}{2}\left[1 + \left(\frac{a}{4}\right) + \left(\frac{a}{4}\right)^2 + \left(\frac{a}{4}\right)^{n-3} + \left(\frac{a}{4}\right)^{n-2}\right]$$

$$= \left(\frac{a}{4}\right)^{n-1} (20) + \frac{1}{2}\left[\frac{1\left(1 - \left(\frac{a}{4}\right)^{n-1}\right)}{1 - \frac{a}{4}}\right]$$

$$= \left(\frac{a}{4}\right)^{n-1} (20) + \frac{1}{2}\left[\frac{4}{4-a}\left(1 - \left(\frac{a}{4}\right)^{n-1}\right)\right]$$

$$= \left(\frac{a}{4}\right)^{n-1} (20) + \frac{2}{4-a} - \frac{2}{4-a}\left(\frac{a}{4}\right)^{n-1}$$

$$= \frac{20(4-a) - 2}{4-a}\left(\frac{a}{4}\right)^{n-1} + \frac{2}{4-a}$$

$$= \frac{78 - 20a}{4-a}\left(\frac{a}{4}\right)^{n-1} + \frac{2}{4-a} \quad \text{(Shown)}$$

(c) For the constant value to be valid, i.e. when the sequence converges, then

$$\left|\frac{a}{4}\right| < 1$$

$$|a| < 4$$

$$-4 < a < 4$$

$\therefore$ the range of a is $-4 < a < 4$.

For $-4 < a < 4$., as $n \to \infty$, $\left(\frac{a}{4}\right)^{n-1} \to 0$.

Therefore, the constant value is $\frac{2}{4-a}$.

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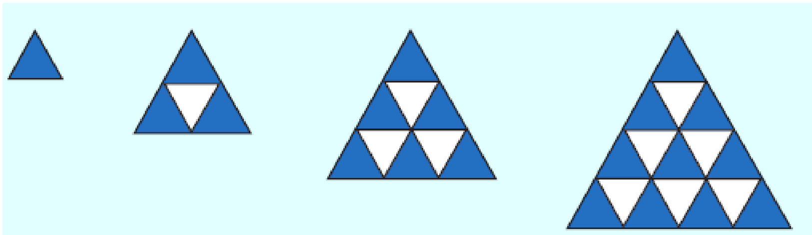
Solution



Refer the above diagram.

1st row: number of matches $= 3$

2nd row: number of matches $= 6$

3rd row: number of matches $= 9$

$\therefore$ nth row: number of matches $= 3 + (n-1)(3)$

$$= 3n$$

1 row: total number of matches $= 3$

2 rows: total number of matches $= 3 + 6$

3 rows: total number of matches $= 3 + 6 + 9$

Total number of matches in n rows

$= \frac{n}{2}\left(\text{number of matches in 1st row} + \text{number of matches in } n\text{th row}\right)$

$= \frac{n}{2}(3 + 3n)$

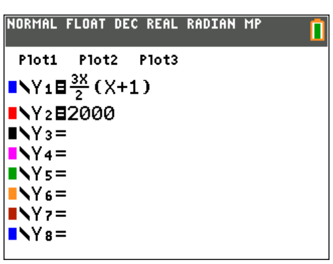
$= \frac{3n(n+1)}{2}$ (Shown)

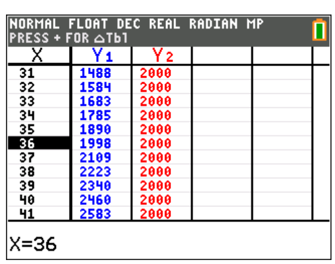
To find the maximum number of complete rows she is able to make with two thousand matchsticks,

set total number of matchsticks used in making a pattern with n rows ≤ 2000

i.e. $\frac{3n(n+1)}{2} \leq 2000$

Using GC,





The maximum number of complete rows she is able to make with two thousand matchsticks is 36.

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Solution

(a) Interest rate for 12 months $= p\%$

Interest rate per month $= \frac{p}{100} \times \frac{1}{12}$

Interest charged their first repayment on 1 September 2021

$= \frac{pL}{1200}$

(b)

Month	Oustanding Loan
0	L
1	$\left(1+\frac{p}{1200}\right)L - x$
2	$\left(1+\frac{p}{1200}\right)\left[\left(1+\frac{p}{1200}\right)L-x\right] - x = \left(1+\frac{p}{1200}\right)^2 L - \left(1+\frac{p}{1200}\right)x - x$
3	$\left(1+\frac{p}{1200}\right)\left[\left(1+\frac{p}{1200}\right)^2 L-\left(1+\frac{p}{1200}\right)x-x\right] - x = \left(1+\frac{p}{1200}\right)^3 L - \left(1+\frac{p}{1200}\right)^2 x - \left(1+\frac{p}{1200}\right)x - x$

Outstanding loan at the start of the nth month after their monthly repayment

$$= \left(1+\frac{p}{1200}\right)^n L - \left(1+\frac{p}{1200}\right)^{n-1} x - \ldots - \left(1+\frac{p}{1200}\right)^2 x - \left(1+\frac{p}{1200}\right)x - x$$

$$= \left(1+\frac{p}{1200}\right)^n L - x\left[\left(1+\frac{p}{1200}\right)^{n-1} + \left(1+\frac{p}{1200}\right)^{n-2} \ldots + 1\right]$$

$$= \left(1+\frac{p}{1200}\right)^n L - x\left[\frac{\left(1+\frac{p}{1200}\right)^n - 1}{\left(1+\frac{p}{1200}\right) - 1}\right]$$

$$= \left(1+\frac{p}{1200}\right)^n L - \frac{1200x}{p}\left[\left(1+\frac{p}{1200}\right) - 1\right] \qquad \text{(Shown)}$$

(c) It is given $L = 504\ 000$, $p = 2.6$ and $n = 360$ (30 years).
For the loan to be repay completely in 30 years,
outstanding loan at the start of the nth month ≤ 0

$$\left(1+\frac{p}{1200}\right)^n L - \frac{1200x}{p}\left[\left(1+\frac{p}{1200}\right)-1\right] \le 0 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (1)$$

Substituting $n = 360$, $p = 2.6$ and $L = 504000$ into (1).

$$504000\left(1+\frac{2.6}{1200}\right)^{360} - \frac{1200x}{2.6}\left(\left(1+\frac{2.6}{1200}\right)^{360} - 1\right) \le 0$$

$$\frac{1200x}{2.6}\left(\left(1+\frac{2.6}{1200}\right)^{360} - 1\right) \ge 504000\left(1+\frac{2.6}{1200}\right)^{360}$$

$$x \ge \frac{1098534.707}{544.4457028}$$

$$\ge 2017.712145$$

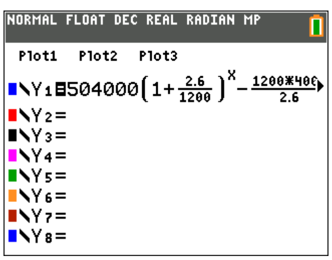
$\therefore$ the monthly repayment is \$2017.71.

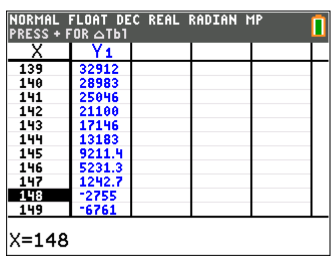
(d) It is given that $x = 4000$, $p = 2.6$ and $L = 504\ 000$.

Substituting $x = 4000$, $p = 2.6$ and $L = 504000$ into (1).

$$504000\left(1+\frac{2.6}{1200}\right)^n - \frac{1200\times 4000}{2.6}\left[\left(1+\frac{2.6}{1200}\right)^n - 1\right] \le 0,$$

Using GC





From the table above, when $n = 148$, $504000\left(1+\dfrac{2.6}{1200}\right)^n - \dfrac{1200\times 4000}{2.6}\left[\left(1+\dfrac{2.6}{1200}\right)^n - 1\right] = -2755 < 0$.

As $n = 148$ months, it is equivalent to 12 years and 4 months.

The date that the couple pays for their final repayment is 1 December 2033.
The final repayment that the couple pays is \$1245.38.

(e) Amount money including interest save after 1 month

$= 1.001(k)$

Amount money including interest save after 2 month

$= 1.001\left[k + a\right]$

Amount money including interest save after 3 month

$= 1.001\left[k + 2a\right]$

Amount money including interest save after n month

$= 1.001\left[k + (n-1)a\right]$

Total amount money including interest save after n month

$= \sum_{r=1}^{n} 1.001\left[k + (n-1)a\right]$

$= 1.001\sum_{r=1}^{n}\left[k + (n-1)a\right]$ .... (1)

Given that total amount money in the saving plan after n months

$= \sum_{r=1}^{n}(450.45 + 50.05r)$

$= 1.001\sum_{r=1}^{n}\left[450 + 50r\right]$

$= 1.001\sum_{r=1}^{n}\left[500 + 50(r-1)\right]$ .... (2)

Comparing (1) and (2)

Hence, $k = 500, a = 50$

Alternative Method

When $n = 1$,

$1.001k = 450.45 + 50.05$

$k = 500$

When $n = 2$,

$1.001k + 1.001(k + a) = 450.45 + 50.05 + 450.45 + 50.45(2)$

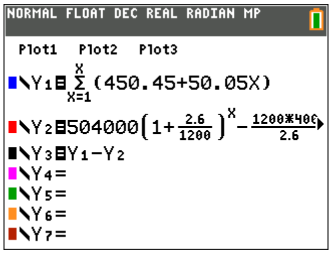
$2k + a = 1050$

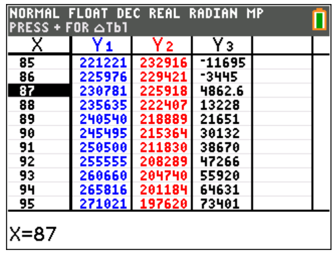
$a = 50$

Hence, $k = 500, a = 50$

(f) $\displaystyle\sum_{r=1}^{n}(450.45+50.05r) \geq 504000\left(1+\frac{2.6}{1200}\right)^{n}-\frac{1200\times 4000}{2.6}\left[\left(1+\frac{2.6}{1200}\right)^{n}-1\right]$

Using GC





$\therefore$ the least number of months is 87

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Solution

(a)(i) Given $S_n = \frac{3}{2} - \frac{(n+1)^2}{2^{n+1}} - \frac{1}{2^n}$

$$\begin{aligned}
u_n &= S_n - S_{n-1} \\
&= \frac{3}{2} - \frac{(n+1)^2}{2^{n+1}} - \frac{1}{2^n} - \left(\frac{3}{2} - \frac{(n-1+1)^2}{2^{n-1+1}} - \frac{1}{2^{n-1}} \right) \\
&= \frac{n^2}{2^n} + \frac{1}{2^{n-1}} - \frac{(n+1)^2}{2^{n+1}} - \frac{1}{2^n} \\
&= \frac{2n^2 + 2^2 - (n+1)^2 - 2}{2^{n+1}} \\
&= \frac{2n^2 + 2 - n^2 - 2n - 1}{2^{n+1}} \\
&= \frac{n^2 - 2n + 1}{2^{n+1}} \\
&= \frac{(n-1)^2}{2^{n+1}} \quad \text{(Shown)}
\end{aligned}$$

(ii)
$$\begin{aligned}
\sum_{r=3}^{n} \frac{(r-1)^2}{2^{r+1}} &= \sum_{r=1}^{n} \frac{(r-1)^2}{2^{r+1}} - \sum_{r=1}^{2} \frac{(r-1)^2}{2^{r+1}} \\
&= \frac{3}{2} - \frac{(n+1)^2}{2^{n+1}} - \frac{1}{2^n} - \frac{1}{8} \\
&= \frac{11}{8} - \frac{(n+1)^2}{2^{n+1}} - \frac{1}{2^n}
\end{aligned}$$

As $n \to \infty, \frac{1}{2^n} \to 0$ and $\frac{(n+1)^2}{2^{n+1}} \to 0$

$\therefore \quad \sum_{r=3}^{\infty} \frac{(r-1)^2}{2^{r+1}} = \frac{11}{8}.$

(b)(i) $(1\times4\times7)+(2\times5\times8)+(3\times6\times9)+\ldots$ to n terms

$$=\sum_{r=1}^{n} r(r+3)(r+6)$$

$$=\sum_{r=1}^{n} r^3+9r^2+18r$$

$$=\sum_{r=1}^{n} r^3+9\sum_{r=1}^{n} r^2+18\sum_{r=1}^{n} r$$

$$=\frac{n^2}{4}(n+1)^2+9\left[\frac{n}{6}(n+1)(2n+1)\right]+18\left[\frac{n}{2}(n+1)\right]$$

$$=\frac{n^2}{4}(n+1)^2+\frac{3n}{2}(n+1)(2n+1)+9n(n+1)$$

$$=\frac{n(n+1)}{4}\left[n(n+1)+6(2n+1)+36\right]$$

$$=\frac{n(n+1)}{4}(n^2+13n+42)$$

$$=\frac{n(n+1)(n+6)(n+7)}{4}$$

(ii) Since every term of $(1\times4\times7)+(2\times5\times8)+(3\times6\times9)+\ldots$ has an even number the sum of first n terms is also an even number.